NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

COURSE MATERIAL

EE 302 ELECTROMAGNETICS

VISION OF THE INSTITUTION
To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered: B.Tech Electrical and Electronics Engineering
- M.Tech (Energy Systems)
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

To excel in technical education and research in the field of Electrical & Electronics Engineering by imparting innovative engineering theories, concepts and practices to improve the production and utilization of power and energy for the betterment of the Nation

DEPARTMENT MISSION

1) To offer quality education in Electrical and Electronics Engineering and prepare the students for professional career and higher studies.

2) To create research collaboration with industries for gaining knowledge about real-time problems.

3) To prepare students with sound technical knowledge

4) To make students socially responsible

PROGRAMME EDUCATIONAL OBJECTIVES

1. Graduates shall have a good foundation in the fundamental and practical aspects of Mathematics and Engineering Sciences so as to build successful and enriching careers in the field of Electrical Engineering and allied areas

2. Graduates shall learn and adapt themselves to the latest technological developments in the field of Electrical & Electronics Engineering which will in turn motivate them to excel in their domains and shall pursue higher education and research

3. Graduates shall have professional ethics and good communication ability along with entrepreneurial skills and leadership skills, so that they can succeed in multidisciplinary and diverse fields.

PROGRAM OUTCOME (PO'S)

Engineering Graduates will be able to:

PO 1. Engineering knowledge: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO 2. Problem analysis: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO 3. Design/development of solutions: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO 4. Conduct investigations of complex problems: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO 5. Modern tool usage: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO 6. The engineer and society: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO 7. Environment and sustainability: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO 8. Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO 9. Individual and team work: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO 10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO 11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO 12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOME(PSO'S)

PSO 1: Apply Science, Engineering, Mathematics through differential and Integral Calculus, Complex Variables to solve Electrical Engineering Problems

PSO 2: Demonstrate proficiency in the use of software and hardware to be required to practice electrical engineering profession.

PSO 3. Apply the knowledge of Ethical and Management principles required to work in a team as well as to lead a team.

Course code	Course Name	L-T-P - Credits	Year of Introduction
EE302	ELECTROMAGNETICS	2-1-0-3	2016

Prerequisite: Nil

Course Objectives

- To develop a conceptual basis of electrostatics, magnetostatics, electromagnetic waves
- To understand various engineering applications of electromagnetics

Syllabus

Introduction to vector calculus, Electrostatics, Electrical potential, energy density and their applications. Magneto statics, magnetic flux density, scalar and vector potential and its applications, Time varying electric and magnetic fields, Electromagnetic waves

Expected outcome .

The students will be able to:

i. Analyze fields and potentials due to static charges
ii. Explain the physical meaning of the differential equations for electrostatic and magnetic fields
iii. Understand how materials are affected by electric and magnetic fields
iv. Understand the relation between the fields under time varying situations
v. Understand principles of propagation of uniform plane waves.
vi. Be aware of electromagnetic interference and compatibility

Text Book:

1. Nannapeni Narayana Rao, "Elements of Engineering Electromagnetics", Prentice Hall India
2. Sadiku M. N. O, *Elements of Electromagnetics*, Oxford university Press, 2010

Data Book (Approved for use in the examination):

References:

1. Cheng D. K., Field and Wave Electromagnetic, Pearson Education, 2013.
2. Edminister J. A., Electromagnetics, Schaum Outline Series , Tata McGraw-Hill, 2006.
3. Gangadhar K. A. and P. M. Ramanathan , Electromagnetic field theory , Khanna Publishers, 2009.
4. Hayt W. H. and J. A. Buck , Engineering Electromagnetics, 8/e, McGraw-Hill, 2012.
5. Inan U. S. and A. S. Inan, Engineering Electromagnetics, Pearson Education, 2010.
6. John Krauss and Daniel A. Fleisch, Electromagnetics with Applications, McGraw-Hill, 5th edition
7. Murthy T. V. S. A, Electromagnetic field, S. Chand Ltd, 2008.
8. Premlet B., Electromagnetic theory with applications, Phasor Books, 2000.
9. S.C.Mahapatra and Sudipta Mahapatra ,Principles of Electromagnetics, McGraw-Hill, 2015

Course Plan

Module	Contents	Hours	Sem. Exam Marks
I	**STATIC ELECTRIC FIELDS:** Introduction to Co-ordinate System – Rectangular – Cylindrical and Spherical Co- ordinate System – Gradient of a Scalar field, Divergence of a Vector field and Curl of a Vector field- Their Physical interpretation. Divergence Theorem, Stokes' Theorem. Numerical problems	6	15%
II	Coulomb's Law, Electric field intensity. Field due to a line charge, Sheet Charge and Continuous Volume Charge distribution. Electric Flux and Flux Density; Gauss's law and its application. Electric Potential-The Potential Gradient. The Electric dipole. The Equipotential surfaces. Capacitance - capacitance of co-axial cable, two wire line. Poisson's and Laplace's equations	8	15%
FIRST INTERNAL EXAMINATION			

III	STATIC MAGNETIC FIELD: Biot-Savart Law, Amperes Force Law.– Magnetic Field intensity due to a finite and infinite wire carrying a current–Magnetic field intensity on the axis of a circular and rectangular loop carrying a current –Magnetic vector potential, Magnetic flux Density and Ampere’s circuital law and simple applications.	6	15%
IV	ELECTRIC AND MAGNETIC FIELDS IN MATERIALS–Electric Polarization-Nature of dielectric materials-Electrostatic energy and energy density–Boundary conditions for electric fields and magnetic fields–Conduction current and displacement current densities–continuity equation for current. Maxwell’s Equation in Differential and integral form from Modified form of Ampere’s circuital law, Faraday's Law and Gauss Law	8	15%
SECOND INTERNAL EXAMINATION			
V	TIME VARYING ELECTRIC AND MAGNETIC FIELDS:. Poynting Vector and Poynting Theorem – Power flow in a co-axial cable – Complex Average Poynting Vector. ELECTROMAGNETIC WAVES: Wave Equation from Maxwell's Equation – Uniform Plane Waves –Wave equation in Phasor form	7	20%
VI	Plane waves propagation in loss less and lossy dielectric medium and conducting medium. Plane wave in good conductor, surface resistance, Skin depth, Intrinsic Impedance and Propagation Constant in all medium. Phase and group velocity. Transmission lines: waves in transmission line –solution for loss less lines –characteristic impedance – VSWR – impedance matching. Introduction to Electromagnetic interference and compatibility.	7	20%
END SEMESTER EXAM			

QUESTION PAPER PATTERN:

Maximum Marks: 100 Exam Duration: 3Hourrs.

Part A: 8 compulsory questions.

One question from each module of Modules I - IV; and two each from Module V & VI.

Student has to answer all questions. (8 x5)=40

Part B: 3 questions uniformly covering Modules I & II. Student has to answer any 2 from the 3 questions: (2 x 10) =20. Each question can have maximum of 4 sub questions (a,b,c,d), if needed.

Part C: 3 questions uniformly covering Modules III & IV. Student has to answer any 2 from the 3 questions: (2 x 10) =20. Each question can have maximum of 4 sub questions (a,b,c,d), if needed.

Part D: 3 questions uniformly covering Modules V & VI. Student has to answer any 2 from the 3 questions: (2 x 10) =20. Each question can have maximum of 4 sub questions (a,b,c,d), if needed.

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SUB CODE: EE302

SUB NAME: ELECTROMAGNETICS

SEM/YEAR: S6/III

CONTENT BEYOND SYLLABUS

Current density and ohms law- electromotive force and Kirchhoff's Law-Equation of continuity and Kirchhoff's current law-power dissipation and joule's law-Boundary condition for current density-Resistance calculations

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NAME & SIGN OF FACULTY

(P.SUNDARAMOORTHI)

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EE 302 ELECTROMAGNETICS

QUESTION BANK- MODULE – I

1) Explain the concept of Cylindrical and Spherical Co- ordinate System –
2) Discuss in details about Gradient of a Scalar field,
3) Write the importance of Divergence of a Vector field
4) Enumerate the Curl of a Vector field- Their Physical interpretation.
5) state and prove the Divergence Theorem,
6) Derive the expression for Stokes ‘theorem

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QUESTION BANK- MODULE – II

1) State and prove the Coulomb's Law,
2) Discuss in details about Electric field intensity and Field due to a line charge,
3) Discuss in details about Electric field intensity due to Sheet Charge
4) Discuss in details about Electric field intensity due to Continuous Volume Charge distribution.
5) Discuss the concept of Electric Flux and Flux Density
6) State and explain Gauss's law and its application.
7) Discuss in details about Electric Potential & Potential Gradient.
8) Enumerate the concept of Electric dipole and the equipotential surfaces.
9) Define Capacitance and derive the expression for capacitance of co-axial cable, two wire line.
10) Derive the expression for Poisson's and Laplace's equations

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QUESTION BANK- MODULE – III

1) State and prove the Biot-Savart Law,
2) State and prove Amperes Force Law.–
3) Derive the expression for Magnetic Field intensity due to a finite and infinite wire carrying a current–
4) Derive the expression for Magnetic field intensity on the axis of a circular and rectangular loop carrying a current
5) Define Magnetic vector potential, Magnetic flux Density
6) Derive the expression for Ampere’s circuital law List the merits and limitations

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QUESTION BANK- MODULE – IV

1) Define Electric Polarization
2) Explain the concept of nature of dielectric materials-
3) Discuss in details about Electrostatic energy and energy density–
4) Derive the expression of Boundary conditions for electric fields and magnetic fields
5) Define Conduction current and displacement current densities
6) Derive the expression of continuity equation for current.
7) Derive the expression of Maxwell’s Equation in Differential and integral form from Modified form of Ampere’s circuital law,
8) Derive the expression for Faraday's Law and Gauss Law

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QUESTION BANK- MODULE – V

1) Derive the expression for Poynting Vector and Poynting Theorem
2) Discuss in details about Power flow in a co-axial cable
3) Define Complex Average Poynting Vector.
4) derive the expression for Wave Equation from Maxwell's Equation
5) Discuss in details about Uniform Plane Waves &
6) Derive the expression for Wave equation in Phasor form

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QUESTION BANK- MODULE – VI

1) Explain the concept of Plane waves propagation in loss less and lossy dielectric medium and conducting medium.
2) Discuss in details about Plane wave in good conductor,& surface resistance,
3) Define Skin depth,
4) Enumerate the importance of Intrinsic Impedance and Propagation Constant in all medium.
5) Define Phase and group velocity.
6) Explain the concept of waves in transmission line & solution for loss less lines
7) Write the concept of characteristic impedance &VSWR
8) Define impedance matching.
9) Discuss in details about Electromagnetic interference and compatibility

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SUB CODE: EE302

SUB NAME: ELECTROMAGNETICS

SEM/YEAR: S6/III

ASSIGNMENT QUESTIONS- (ASSIGNMENT - I)

Answer All Questions (3*10=30 MARKS)

1. Explain in details about divergence theorem
2. Discuss the magnetic field intensity on the axis of rectangular coil in detail
3. Derive the expression for gauss's law in details

NAME & SIGN OF FACULTY

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SUB CODE: EE302

SUB NAME: ELECTROMEGNETICS

SEM/YEAR: S6/III

ASSIGNMENT QUESTIONS- (ASSIGNMENT - II)

Answer All Questions (3*10=30 MARKS)

1. Derive the expression for Maxwell's equation in differential & rectangular forms of amperes circuit law
2. Discuss in details about poynthing vector &theorem
3. With neat diagram explain in detail about surface resistance and skin depth.

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Module 1

Coordinate System and Vector Calculus

COORDINATE SYSTEM AND TRANSFORMATION

Ques 1) What do you mean by scalar and vector quantities?

Or

What is unit vector? Also define scalar and vector field.

Ans: Scalar Quantities

A scalar quantity is defined as a quantity that has magnitude only no direction. Typical **examples** of scalar quantities are time, speed, temperature, and volume.

Vector Quantities

A vector quantity is defined as a quantity that has both magnitude and direction. Typical **examples** of vector quantities are displacement, velocity, acceleration, momentum and force.

Scalar Field

A field is a region in which a particular physical function has a value at each and every point in that region. The distribution of a scalar quantity with a definite position in a space is called **scalar field. For example,** the temperature of atmosphere, it has a definite value in the atmosphere but no need of direction to specify it hence it is a scalar field.

Vector Field

If a quantity which is specified in a region to define a field is a vector then the corresponding field is called a **vector field. For example,** the gravitational force on a mass in a space is a vector field. This force has a value at various points in a space and always has a specific direction.

In case if a vector is known then the unit vector along that vector can be obtained by dividing the vector by its magnitude. Thus **unit vector** can be expressed as, direction a unit vector can be used.

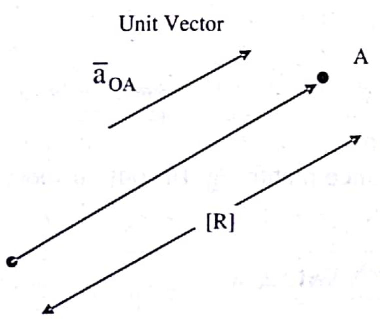

Figure 1.1: Unit Vector

Consider a unit vector $\bar{a}_{OA}$ in the direction of $\overline{OA}$ as shown in the **figure 1.1**. This vector indicates the direction of $\overline{OA}$ but its magnitude is unity.

Unit vector can be represented as:

$$\text{Unit Vector}\,\bar{a}_{OA} = \frac{\overline{OA}}{|\overline{OA}|}$$

Any vector A is expressed in two forms:

1) $A = (A_x, A_y, A_z)$. where, A_x, A_y, A_z are known as the components of vector A.
2) $A = A_x a_x + A_y a_y + A_z a_z$. where, a_x, a_y, a_z are unit vectors along the coordinate axis.

The magnitude of A is written as A,

i.e., $A = |A|$

The unit vector of A is '$\hat{a}$' and it is given by,

$$\hat{a} = \frac{\bar{A}}{|A|}$$

Ques 2) Discuss the addition, subtraction and multiplication of vectors.

Or

What do you mean by scalar product and vector product?

Or

Explain the following operations performed on vectors:
1) **Addition and subtraction (Sum and difference)**
2) **Dot product and cross product /Multiplication of Vectors**
3) **Triple product**

Ans: Addition and Subtraction / Sum and Difference of Vectors

The addition and subtraction or the sum and difference of two vectors are given by,

$A + B = (A_x + B_x)\, a_x + (A_y + B_y)\, a_y + (A_z + B_z)\, a_z$

$A - B = (A_x - B_x)\, a_x + (A_y - B_y)\, a_y + (A_z - B_z)\, a_z$

Multiplication of Vectors

The multiplication of vectors is of three types:

1) **Dot/Scalar Product:** The dot product is denoted by A.B or B.A and given by,

   $A.B = B.A = AB \cos\theta$

   $= A_x B_x + A_y B_y + A_z B_z$

   Here θ is the angle between the vectors A and B. Dot product of two vectors is a scalar.

2) **Cross/Vector Product:** The cross product is denoted by $A \times B$ and given by,
$A \times B = AB \sin\theta\, a_n$.

Where a_n is the unit vector perpendicular to A and B

$$\text{Or}\quad A \times B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= a_x [A_y B_z - A_z B_y] + a_y [A_z B_x - A_x B_z] + a_z [A_x B_y - A_y B_x]$$

Where, $A = A_x a_x + A_y a_y + A_z a_z$ and
$B = B_x a_x + B_y a_y + B_z a_z$

The cross product of two vectors is a vector.

3) **Triple Products:** Multiplication of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is called vector triple product. The product of three vectors is classified into two categories:

i) **Scalar Triple Product:** For the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, scalar triple product is defined as,
$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$$

Since the result is a scalar quantity, this is known a **scalar triple product.**

If the components of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are given as $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$, $\vec{C} = (C_x, C_y, C_z)$, respectively, then the scalar triple product is obtained by the determinant of a 3 × 3 matrix given as,

$$\vec{A}.(\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

ii) **Vector Triple Product:** For the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, vector triple product is defined as,
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}.\vec{C}) - \vec{C}.(\vec{A}.\vec{B})$$

Ques 3) If $A = 2a_x + 5a_y + 6a_z$ and $B = a_x - 3a_y + 6a_z$, find A + B and A – B.

Ans: $A + B = (A_x + B_x)\, a_x + (A_y + B_y)\, a_y + (A_z + B_z)\, a_z$
$= (2 + 1)\, a_x + (5 - 3)\, a_y + (6 + 6)\, a_z$
$A + B = 3a_x + 2a_y + 12a_z$
$A - B = (2 - 1)\, a_x + (5 + 3)\, a_y + (6 - 6)\, a_z$
$A - B = a_x + 8a_y$

Ques 4) If $A = a_x + a_y + 2a_z$ and $B = 2a_x + a_y + a_z$, find A.B.

Ans: $A.B = A_x B_x + A_y B_y + A_z B_z$
$= 1.2 + 1.1 + 2.1$
$= 2 + 1 + 2$
$= 5$

Ques 5) Given $A = 2a_x + a_y + 2a_z$ and $B = a_x + 2a_y + a_z$, find A × B.

$$\textbf{Ans:}\quad A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$
$$= a_x (1 - 4) + a_y (2 - 2) + a_z (4 - 1)$$

$A \times B = -3a_x + 3a_z$

Ques 6) For A = (1, 3, 4) and B = (1, 0, 2), find A.B.

Ans: Here
$A_x = 1, A_y = 3, A_z = 4$
$B_x = 1, B_y = 0, B_z = 2$
$A.B = A_x B_x + A_y B_y + A_z B_z$
$= 1 \times 1 + 3 \times 0 + 4 \times 2$
$= 1 + 0 + 8 = 9$

Ques 7) What do you understand by coordinate systems? Discuss various types.

Or

Describe the following coordinate systems:
1) **Rectangular or Cartesian Coordinates**
2) **Cylindrical or Circular Coordinates**
3) **Spherical or Polar Coordinates**

Ans: Coordinate Systems

Coordinate system is defined as a system to describe uniquely the spatial variation of a quantity at all points in space. There are three types of coordinate system as described below:

1) **Rectangular or Cartesian Coordinates (x, y, z):** A point P in Cartesian coordinates is represented as P (x, y, z). The ranges of coordinate variables are:
$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty \qquad .....(1)$$

From **figure 1.2 (b)**, it is understood that any point in rectangular coordinates is the intersection of three planes:
i) Constant x-plane,
ii) Constant y-plane, and
iii) Constant z-plane, which are mutually perpendicular.

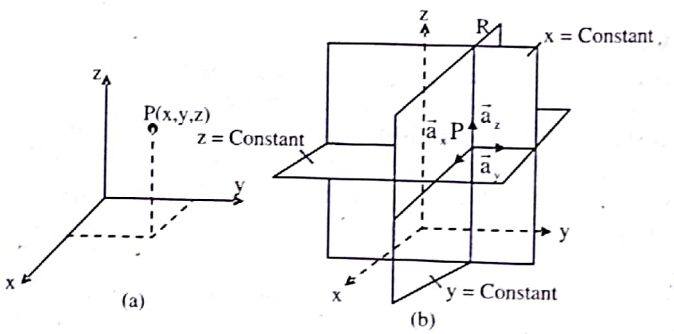

Figure 1.2: (a) Cartesian Coordinates, and (b) Constant x, y, z Planes

A vector $\vec{A}$ in Cartesian coordinate system is written as
$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \qquad ..... (2)$$

Where, $\hat{a}_x, \hat{a}_y, \hat{a}_z$ are the unit vectors along the x, y and z directions, respectively.

From the definitions of dot product, we see that

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1$$

$$\hat{a}_x.\hat{a}_y = \hat{a}_y.\hat{a}_z = \hat{a}_z.\hat{a}_x = 0 \qquad \text{..... (3)}$$

From the definitions of cross product, we see that

$$\hat{a}_x \times \hat{a}_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z ; \hat{a}_y \times \hat{a}_z = \hat{a}_x , \hat{a}_z \times \hat{a}_x = \hat{a}_y \qquad \text{.....(4)}$$

2) **Cylindrical or Circular Coordinates (r, φ, z):** A point P in cylindrical coordinates is represented as P(r, φ, z).

Here, r = Radius of the cylinder passing through

φ = Angle measured from the x-axis in the xy-plane, known as azimuthal angle

z = same as in Cartesian coordinates

The ranges of coordinate variables are

$$0 \le r < \infty$$

$$0 \le \phi < 2\pi$$

$$-\infty < z < \infty \qquad \text{.....(5)}$$

From **figure 1.3 (b)**, it is understood that any point in cylindrical coordinates is an intersection of three planes *viz.*,

i) Constant 'r' plane (a circular cylinder),
ii) Constant φ plane (semi-infinite plane with its edge along the z-axis, and
iii) Constant z-plane (parallel to xy-plane).

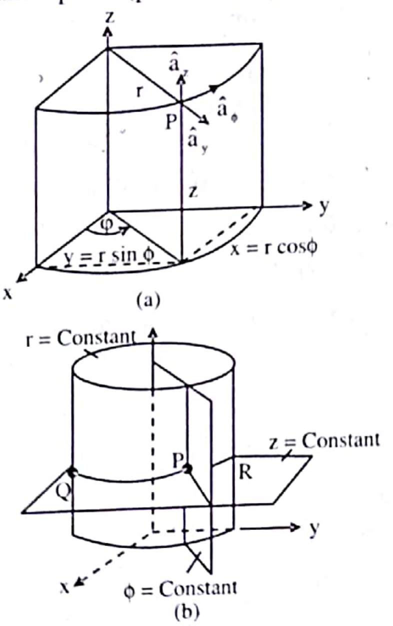

Figure 1.3: (a) Cylindrical Coordinates, and (b) Constant, r, φ, z Planes

A vector $\vec{A}$ in cylindrical coordinate system is written as,

$$\vec{A} = A_r\hat{a}_r + A_\phi\hat{a}_\phi + A_z\hat{a}_z \qquad \text{..... (6)}$$

Where $\hat{a}_r, \hat{a}_\phi, \hat{a}_z$ are the unit vectors along the r, φ and z directions, respectively.

From the definitions of dot product, we see that

$$\hat{a}_r.\hat{a}_r = \hat{a}_\phi.\hat{a}_\phi = \hat{a}_z.\tilde{a}_z = 1$$

$$\hat{a}_r.\hat{a}_\phi = \hat{a}_\phi.\hat{a}_z = \hat{a}_z.\hat{a}_r = 0 \qquad \text{..... (7)}$$

From the definitions of cross product, we see that

$$\hat{a}_r \times \hat{a}_r = \hat{a}_\phi \times \hat{a}_\phi = \hat{a}_z \times \hat{a}_z = 0$$

$$\hat{a}_r \times \hat{a}_\phi = \hat{a}_z ; \hat{a}_\phi \times \hat{a}_z = \hat{a}_r ; \hat{a}_z \times \hat{a}_r = \hat{a}_\phi \qquad \text{.....(8)}$$

3) **Spherical or Polar Coordinates (ρ, θ, φ):** A point P in spherical coordinates is represented as P (ρ, θ, φ)

Here,

ρ = Distance of the point from the origin,

r = Radius of a sphere centered at the origin and passing through the point P

θ = Angle between the z-axis and the position vector P, known as colatitudes

φ = Angle measured from the x-axis in the xy-plane, known as azimuthal angle (Same as in cylindrical coordinates).

The ranges of coordinate variables are

$$0 \le \rho < \infty$$

$$0 \le \theta < \pi \qquad \text{.....(9)}$$

$$0 \le \phi < 2\pi$$

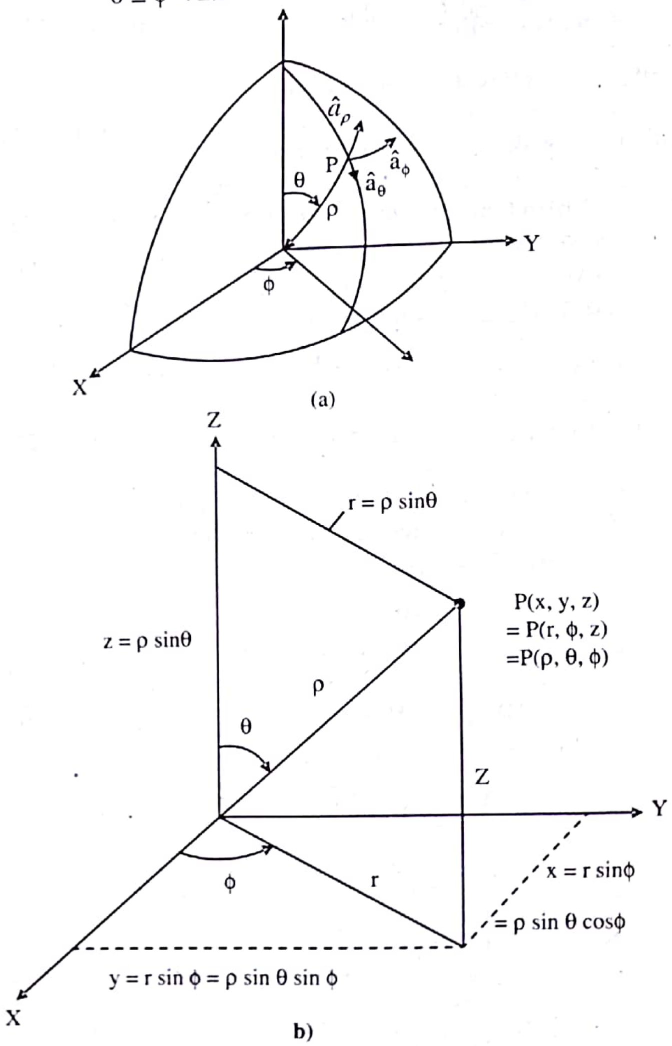

Figure 1.4: (a) Constant ρ, θ, φ Planes, and (b) Point P and Unit Vectors in Spherical Coordinates

A vector $\vec{A}$ in spherical coordinate system is written as,

$$\bar{A} = A_\rho\hat{a}_\rho + A\hat{a} + A_\phi\hat{a}_\phi \qquad \text{..... (10)}$$

Where, $\hat{a}_\rho$, $\hat{a}_\theta$, $\hat{a}_\phi$ are the unit vectors along the ρ, θ and ϕ directions, respectively.

From the definitions of dot product, we see that

$$\hat{a}_\rho.\hat{a}_\rho = \hat{a}_\theta.\hat{a}_\theta = \hat{a}_\phi.\hat{a}_\phi = 1$$

$$\hat{a}_\rho.\hat{a}_\theta = \hat{a}_\theta.\hat{a}_\phi = \hat{a}_\phi.\hat{a}_\rho = 0 \quad ..... (11)$$

From the definitions of cross product, we see that

$$\hat{a}_\rho \times \hat{a}_\rho = \hat{a}_\theta \times \hat{a}_\theta = \hat{a}_\phi \times \hat{a}_\phi = 0$$

$$\hat{a}_\rho \times \hat{a}_\theta = \hat{a}_\phi \,;\, \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_\rho \,;\, \hat{a}_\phi \times \hat{a}_\rho = \hat{a}_\theta \quad ...(12)$$

Ques 8) What do you mean by coordinate transformation? Also give the relation between coordinate systems.

Or

Transform the Cartesian coordinate system into Spherical and Cylindrical coordinate system.

Or

Convert the Cartesian coordinate system into cylindrical coordinate system.

Ans: Coordinate Transformation

Conversion of one coordinate system into other system is called **coordinate transformation**. Transformations of various coordinate systems are described below:

1) **Transformation of Cartesian (x, y, z) to Cylindrical (r, ϕ, z) Coordinates:** The relationship between Cartesian (x, y, z) and cylindrical (r, ϕ, z) coordinates can be obtained and written as,

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right), \qquad z = z \quad ......(1)$$

$$\text{and } x = r\cos\phi,\ y = r\sin\phi,\ z = z \quad ......(2)$$

The relationships between the unit vectors are obtained from **figure 1.5** and are given as,

$$\hat{a}_x = \cos\phi\ \hat{a}_r - \sin\phi\,\hat{a}_\phi$$

$$\hat{a}_y = \sin\phi\ \hat{a}_r - \cos\phi\ \hat{a}_\phi \quad ..... (3)$$

$$\hat{a}_z = \hat{a}_z \text{ and } \hat{a}_r = \cos\phi\ \hat{a}_x + \sin\phi\,\hat{a}_y$$

$$\hat{a}_\phi = -\sin\phi\ \hat{a}_x + \cos\phi\,\hat{a}_y \quad ..... (4)$$

$$\hat{a}_z = \hat{a}_z$$

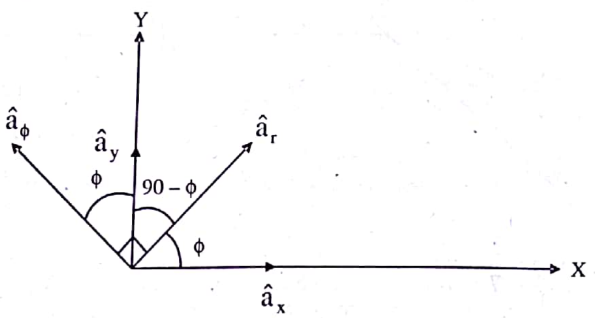

Figure 1.5: Unit Vector Transformation between Cartesian and Cylindrical Coordinates

The relationship between the component vectors (A_x, A_y, A_z) and (A_r, A_ϕ, A_z) are obtained by using equations (3) and (4) and then rearranging the terms. This is given as,

$$\vec{A} = (A_x\cos\phi + A_y\sin\phi)\,\hat{a}_r + (-A_x\sin\phi + A_y\cos\phi)\hat{a}_\phi + A_z\hat{a}_z$$

$$= (A_r\cos\phi - A_\phi\sin\phi)\,\hat{a}_x + (A_r\sin\phi + A_\phi\cos\phi)\hat{a}_y + A_z\hat{a}_z \quad ..... (5)$$

Thus, the relationships between the component vectors can be written in matrix forms as,

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\text{and } \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \quad .....(6)$$

2) **Transformation of Cartesian (x, y, z) to Spherical (ρ, θ, ϕ) Coordinates:** The relationships between Cartesian (x, y, z) and spherical (ρ, θ, ϕ) coordinates can be written as,

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right),$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \quad .....(7)$$

$$\text{and } x = \rho\sin\theta\cos\phi,$$

$$y = \rho\sin\theta\sin\phi,\ z = \rho\cos\theta \quad .....(8)$$

The relationships between the unit vectors are obtained from **figure 1.6** and are given as,

$$\hat{a}_x = \sin\theta\cos\phi\hat{a}_\rho + \cos\theta\cos\phi\hat{a}_\theta - \sin\phi\hat{a}_\phi$$

$$\hat{a}_y = \sin\theta\sin\phi\hat{a}_\rho + \cos\theta\sin\phi\hat{a}_\theta + \cos\phi\hat{a}_\phi \quad ....(9)$$

$$\hat{a}_z = \cos\theta\hat{a}_\rho - \sin\theta\hat{a}_\theta$$

$$\text{And, } \hat{a}_\rho = \sin\theta\cos\phi\hat{a}_x + \sin\theta\sin\phi\hat{a}_y + \cos\theta\hat{a}_z$$

$$\hat{a}_\theta = \cos\theta\cos\phi\hat{a}_x + \cos\theta\sin\phi\hat{a}_y - \sin\theta\hat{a}_z \quad .....(10)$$

$$\hat{a}_\phi = \sin\phi\hat{a}_x + \cos\phi\hat{a}_y$$

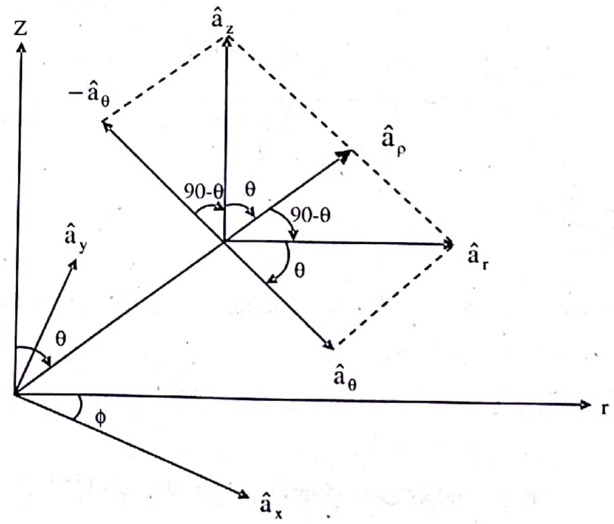

Figure 1.6: Unit Vector Transformation for Cartesian and Spherical Coordinates

The relationships between the component vectors (A_x, A_y, A_z) and (A_ρ, A_θ, A_ϕ) can be obtained by using equations (9) and (10) and then rearranging the terms. This is written in matrix form as,

$$\begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{..... (11)}$$

And,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} \quad \text{..... (12)}$$

3) **Transformation of Cylindrical (r, ϕ, z) to Spherical (ρ, θ, ϕ) Coordinates:** The relationships between cylindrical (r, ϕ, z) and spherical (ρ, θ, ϕ) coordinates are written as,

$$\rho = \sqrt{r^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{r}{z}\right), \qquad \phi = \phi \quad \text{..... (13)}$$

And, $r = \rho \sin\theta \quad \phi = \phi \quad z = \rho\cos\theta$ .....(14)

The relationships between the unit vectors are obtained from **figure 1.7** and are given as,

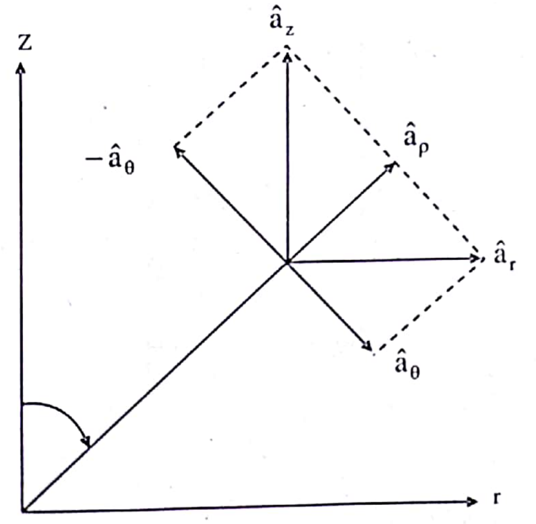

Figure 1.7: Unit Vector Transformation for Cylindrical and Spherical Coordinates

$$\begin{aligned} \hat{a}_\rho &= \sin\theta\,\hat{a}_r + \cos\theta\,\hat{a}_z \\ \hat{a}_\theta &= \cos\theta\cos\phi\,\hat{a}_r - \sin\theta\,\hat{a}_z \\ \hat{a}_\phi &= \hat{a}_\phi \end{aligned} \quad \text{..... (15)}$$

And,
$$\begin{aligned} \hat{a}_r &= \sin\theta\,\hat{a}_\rho + \cos\theta\cos\phi\,\hat{a}_\theta \\ \hat{a}_\phi &= \hat{a}_\phi \\ \hat{a}_z &= \cos\theta\,\hat{a}_\rho - \sin\theta\,\hat{a}_\theta \end{aligned} \quad \text{..... (16)}$$

The relationships between the component vectors (A_ρ, A_θ, A_ϕ) and (A_r, A_ϕ, A_z) can be obtained by using equations (15) and (16) and then rearranging the terms. This is written in matrix form as,

$$\begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} \quad \text{..... (17)}$$

$$\text{And} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} \quad \text{.....(18)}$$

Ques 9) Transform Vector $A = y\hat{a}_x + (x+z)\hat{a}_y$ into spherical coordinates system. Also evaluate at p (–2, 6, 3).

Ans: In spherical form

$$\begin{bmatrix} A_r \\ A_\theta \\ A_a \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

Or, $A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$

$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$

$A_\phi = -y\sin\phi + (x+z)\cos\phi$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$

Substituting these yields:

$A = (A_r, A_\theta, A_\phi) = r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]a_r$
$+ r[\sin\theta\cos\theta\sin\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]a_\theta + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]a_\phi$

At point P(–2, 6, 3)

$x = -2, y = 6, z = 3$

Hence, $e = \sqrt{x^2 + y^2} = \sqrt{4+36} = 6.32$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2}$$

$$z = 3$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{2} = \tan^{-1}\frac{\sqrt{40}}{3}$$

And, $\cos\phi = \dfrac{-2}{\sqrt{40}}$, $\sin\phi = \dfrac{6}{\sqrt{40}}$

$\cos\phi = \dfrac{3}{7}$ $\sin\theta = \dfrac{\sqrt{40}}{7}$

$$A = 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]a_r$$
$$+7\left[\frac{\sqrt{40}}{7}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]a_\theta$$

$$+7\left[\frac{-\sqrt{40}}{7}\cdot\frac{36}{40}+\left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}}+\frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]a_\phi$$

$$=\frac{-6}{7}a_r-\frac{18}{7\sqrt{40}}a_\theta-\frac{38}{\sqrt{40}}a_\phi$$

$$=-0.8571a_r-0.4066a_\theta-6.008a_\phi$$

Note: |A| is the same in the three systems; i.e., |A(x, y, z)| = |A(e, ϕ, z)| = |A(r, θ, ϕ)| = 6.083

Ques 10) Transform the point P (1, 1, 6) in spherical coordinate system.

Ans: At point P, x = 1 y = 1 z = 6

The spherical coordinates are (r, θ, ϕ)

$$\because\ r=\sqrt{x^2+y^2+z^2}=\sqrt{1^2+1^2+36^2}=\sqrt{38}=6.16$$

$$\theta=\tan^{-1}\frac{\sqrt{x^2+y^2}}{x}=\tan^{-1}\frac{\sqrt{1+1}}{6}=13.26°$$

$$\phi=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{1}{1}=45°$$

Hence the spherical coordinates are (6.16, 13.26°, 45°).

Ques 11) Express $B=\left(\frac{10}{r}\right)r+r\cos\theta\cdot\theta$ in cylindrical coordinates.

Ans: For spherical to cylindrical vector transformation

$$\begin{bmatrix}B_\rho\\B_\phi\\B_z\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta & 0\\0 & 0 & 1\\\cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\frac{10}{r}\\r\cos\theta\\1\end{bmatrix}$$

or $B_\rho=\frac{10}{r}\sin\theta+r\cos^2\theta$

$B_\phi=1$

$B_z=\frac{10}{r}\cos\theta-r\sin\theta\cos\theta$

But $r=\sqrt{\rho^2+z^2}$ and $\theta=\tan^{-1}\frac{\rho}{z}$

Thus $\sin\theta=\frac{\rho}{\sqrt{\rho^2+z^2}}$ $\cos\theta=\frac{Z}{\sqrt{\rho^2+z^2}}$

$$B_\rho=\frac{10\rho}{\rho^2+z^2}+\sqrt{\rho^2+z^2}\cdot\frac{z^2}{\rho^2+z^2}$$

$$B_z=\frac{10z}{\rho^2+z^2}-\sqrt{\rho^2+z^2}\cdot\frac{\rho_z}{\rho^2+z^2}$$

Hence,

$$B=\left(\frac{10\rho}{\rho^2+z^2}+\frac{z^2}{\sqrt{\rho^2+z^2}}\right)\bar{a}_\rho+\bar{a}_\phi+\left(\frac{10z}{\rho^2+z^2}-\frac{\rho_z}{\sqrt{\rho^2+z^2}}\right)\bar{a}_z$$

VECTOR CALCULUS

Ques 12) Determine the differential length, area and volume for Cartesian, spherical and cylindrical coordinate system.

Or

Obtain the elemental displacement, area and volume for spherical and cylindrical coordinate systems.

Ans: Differential Length, Area and Volume for Coordinate Systems

To obtain the differential elements in length, area and volume, we consider the following **figures 1.8:**

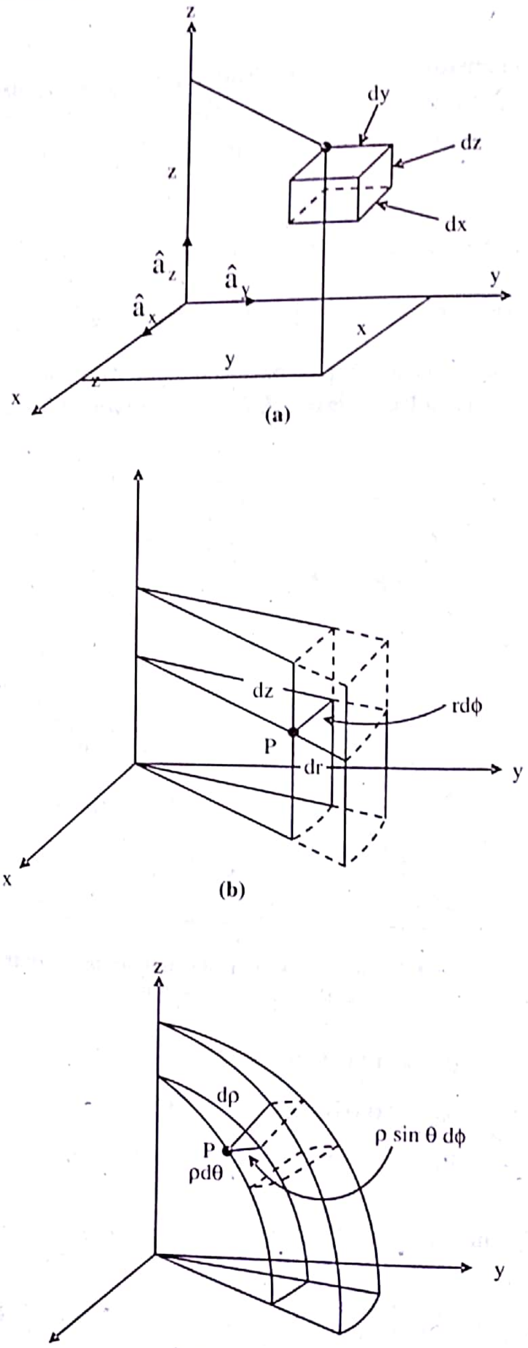

Figure 1.8: (a) Differential Elements in Cartesian Coordinates, (b) Differential Elements in Cylindrical Coordinates and (c) Differential Elements in Spherical Coordinates

These relations are given in **table 1.1.**

Table 1.1: Differential Elements in Different Coordinate Systems

Differential Elements	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Length	$d\vec{s} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$	$d\vec{s} = dr\,\hat{a}_r + r\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$	$d\vec{s} = d\rho\,\hat{a}_\rho + \rho\,d\theta\,\hat{a}_\theta + \rho\sin\theta\,d\phi\,\hat{a}_\phi$
Area	$d\vec{S} = dy\,dz\,\hat{a}_x + dx\,dz\,\hat{a}_y + dx\,dy\,\hat{a}_z$	$+dr\,dz\,\hat{a}_\phi$ $d\vec{S} = r\,d\phi\,dz\,\hat{a}_r$ $+ r\,dr\,d\phi\,\hat{a}_z$	$d\vec{S} = \rho^2\sin\theta\,d\theta\,d\phi\,\hat{a}_\rho$ $+\rho\sin\theta\,d\rho\,d\phi\,\hat{a}_\theta$ $+\rho\,d\rho\,d\theta\,\hat{a}_\phi$
Volume	$dV = dx\,dy\,dz$	$dV = r\,dr\,d\phi\,dz$	$dV = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi$

Ques 13) What do you mean by vector integration? Also discuss the line, surface and volume integral.

Ans: Vector Integration or Vector Integrals

The integration of a vector may be obtained in three ways i.e., line integral, surface integral and volume integral as discussed below:

1) **Line Integral:** The line integral of a vector is the integral of the dot product of the vector and the differential length vector tangential to a specified path.

For vector $\vec{F}$ and a path ℓ, the line integral is given by,

$$\int_{\ell} \vec{F}.\overrightarrow{d\ell} = \int_a^b |\vec{F}| \cos\theta\, d\ell \qquad \text{..... (1)}$$

2) **Surface Integral:** For a vector $\vec{F}$, continuous in a region containing a smooth surface S, the surface integral or the flux of $\vec{F}$ through S is defined as,

$$\psi = \int_S \vec{F}.d\vec{S} = \int_S \vec{F}.\hat{a}_n dS = \int_S |\vec{F}|\cos\theta\, dS \qquad \text{..... (2)}$$

Where, $\hat{a}_n$ = Unit normal vector to the surface S.

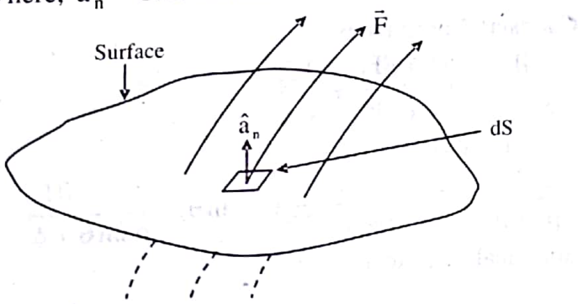

Figure 1.9: Surface Integral

3) **Volume Integral:** The volume integral of a scalar quantity F over a volume V is written as,

$$U = \int_V F dv \qquad \text{..... (3)}$$

The concept of volume integral is necessary to calculate the charge or mass of an object, which are distributed in the volume.

Ques 14) Given that $D = \left(\frac{5r^2}{4}\right) r$ in spherical coordinate. Find the volume enclosed between r = 1, r = 2.

Ans: $V = \int \nabla \cdot D\, dv$ (volume enclosed)

$dv = r^2 \sin\theta\, d\theta\, d\phi\, dr$

$$\nabla \cdot D = \frac{1}{r^2}\left[\frac{\partial r^2}{\partial r}\left(\frac{5r^2}{4}\right)\right]$$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(\frac{5r^4}{4}\right)\right] = \frac{1}{r^2} \times \frac{20 \cdot r^3}{4} = 5r$$

$$V = \int (\nabla \cdot D) dV$$

$$= 5r \cdot r^2 \sin\theta \cdot d\theta\, d\phi \cdot dr$$

$$= 5\int_1^2 r^3 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= 5\left[\frac{r^4}{4}\right]_1^2 \cdot [-\cos\theta]_0^{\pi} \times 2\pi$$

$$= \frac{5}{4}[16-1] \times 2 \times 2\pi = \frac{5 \times 15}{4} \times 4\pi = 75\pi$$

Ques 15) What do you mean by Del operator?

Ans: Del Operator or Differential Vector Operator (∇)

The differential vector operator (∇) or Del or Nabla, in Cartesian coordinates, is defined as,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \qquad \text{..... (1)}$$

This Del is merely a vector operator but not a vector quantity. When it operates on a scalar function, a vector is created. Since a vector, in general, is a function of a space and times both, the del operator is a vector space function operator. It is defined in terms of partial derivatives with respect to space.

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}\hat{a}_3 \qquad \text{..... (2)}$$

For the different coordinate systems, values of h_1, h_2, and h_3 are given in the **table 1.2** below.

Table 1.2

Coordinate System	h_1	h_2	h_3
Cartesian System	1	1	1
Cylindrical System	1	ρ	1
Spherical System	1	r	r sin θ

Substituting the values of h_1 in different coordinate systems, we obtain the relation of del in three different coordinate systems as,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \text{ (Cartesian coordinates)} \quad ..... (3)$$

$$= \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \text{ (Cylindrical coordinates)} \quad .... (4)$$

$$= \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

(Spherical coordinates) ..... (5)

Ques 16) Describe the gradient of a scalar field, divergence and curl of a vector field. Also give its physical interpretation or physical significance.

Or

Define the values of divergence, gradient and curl for spherical and cylindrical coordinate systems.

Or

Explain the physical significance of divergence and curl.

Ans: Gradient of Scalar Field

Gradient of a scalar is a vector and is defined as,

$$\nabla V = \frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z$$

Examples are gradient of temperature, gradient of electric potential and so on.

It gives the maximum space rate of change of the scalar. The scalar can be temperature, potential and so on.

Physical Interpretation/Physical Significance

The gradient of a scalar quantity is the maximum space rate of change of the function.

For example, Let us consider a room in which the temperature is given by a scalar field T, so at any point (x, y, z) the temperature is T(x, y, z) (assuming that the temperature does not change with time). Then, at any arbitrary point in the room, the gradient of T indicates the direction in which the temperature rises most rapidly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

$$\nabla F = \frac{1}{h_1}\frac{\partial F}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial F}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial F}{\partial u_3}\hat{a}_3 \quad ..... (7)$$

$$\nabla F = \frac{\partial F}{\partial x}\hat{a}_x + \frac{\partial F}{\partial y}\hat{a}_y + \frac{\partial F}{\partial z}\hat{a}_z \quad .....(8)$$

(Cartesian Coordinates)

$$= \frac{\partial F}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial F}{\partial \phi}\hat{a}_\phi + \frac{\partial F}{\partial z}\hat{a}_z \quad ..... (9)$$

(Cylindrical Coordinates)

$$= \frac{\partial F}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial F}{\partial \theta}\hat{a}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial F}{\partial \phi}\hat{a}_\phi \quad ..... (10)$$

(Spherical Coordinates)

Divergence of Vector Field

Divergence of a vector at any point is defined as the limit of its surface integral per unit volume as the volume enclosed by the surface around the point shrinks to zero.

$$\therefore \text{div}\vec{F} = \nabla.\vec{F} = \lim_{v\to 0}\left(\frac{\oint_S \vec{F}.d\vec{S}}{v}\right) = \lim_{v\to 0}\left(\frac{\oint_S \vec{F}.\hat{a}_n dS}{v}\right) \quad ..... (1)$$

Where, v is the volume of a arbitrarily shaped region in space that includes the point, S is the surface of that volume, and the integral is a surface integral with $\hat{a}_n$ being the outward normal to that surface.

By definition, this is the divergence of the vector.

$$\therefore \quad \text{div } \vec{F} = \nabla.\vec{F} = \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) \quad ..... (2)$$

Physical Interpretation/Physical Significance

The physical interpretation or significance of the divergence of a vector field is the rate at which the density of a vector exits a given region of space.

For example, we consider air as it is heated or cooled. The relevant vector field for this example is the velocity of the moving air at a point. If air is heated in a region, it will expand in all directions such that the velocity field points outward from that region. Therefore, the divergence of the velocity field in that region would have a positive value, as the region is a source. If the air cools and contracts, the divergence is negative and the region is called a sink.

By definition, this is the divergence of the vector.

$$\nabla.\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_1h_3) + \frac{\partial}{\partial u_3}(F_3h_1h_2)\right] \quad ..... (3)$$

By substituting the value of h_1, h_2, h_3 from **table 1.2** i.e.,

For Cartesian Coordinates $h_1=h_2=h_3=1$

For Cylindrical Coordinates $h_1=1$, $h_2=\rho$, $h_3=1$

For Cylindrical Coordinates $h_1=1$, $h_2=r$, $h_3=r\sin\theta$

The relation of divergence in three different coordinate systems is given as,

$$\nabla.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad ..... (4)$$

(Cartesian Coordinates)

$$= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad ..... (5)$$

(Cylindrical coordinates)

$$= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \theta}(F_\theta \sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial F_\phi}{\partial \phi} \quad ..... (6)$$

(Spherical coordinates)

Curl of Vector Field

It is defined as the limit of the ratio of the integral of the cross product of the vector with outward drawn normal over a closed surface, to the volume enclosed by the surface, as the volume tends to zero.

Mathematically can be written as,

$$\text{Curl}\,\vec{F} = \lim_{v\to 0}\left(\frac{\oint_S \vec{F}\times\hat{a}_n dS}{v}\right) = \lim_{v\to 0}\left(\frac{\oint_S \vec{F}\times d\vec{S}}{v}\right) \quad ..... (11)$$

In matrix form, this can be written as,

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix} \quad .....(12)$$

Physical Interpretation/Physical Significance

The physical interpretation or significance of the curl of a vector at any point is that it provides a measure of the amount of rotation or angular momentum of the vector around the point.

Or in matrix form,

$$\nabla\times\vec{F}=\left(\frac{1}{h_1h_2h_3}\right)\begin{vmatrix}h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3\\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\ F_1h_1 & F_2h_2 & F_3h_3\end{vmatrix} \quad .....(13)$$

By substituting the value of h_1, h_2, h_3 from **table 1.2** i.e.,
For Cartesian Coordinates $h_1=h_2=h_3=1$
For Cylindrical Coordinates $h_1=1$, $h_2=\rho$, $h_3=1$
For Cylindrical Coordinates $h_1=1$, $h_2=r$, $h_3=r\sin\theta$

Or in matrix form as,

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix} \text{ (Cartesian Coordinates) ...(14)}$$

$$=\left(\frac{1}{r}\right)\begin{vmatrix}\hat{a}_r & r\hat{a}_\phi & \bar{a}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ F_r & rF_\phi & F_z\end{vmatrix} \text{ (Cylindrical Coordinates) ...(15)}$$

$$=\left(\frac{1}{\rho^2\sin\theta}\right)\begin{vmatrix}\bar{a}_\rho & \rho\bar{a}_\theta & \rho\sin\theta\bar{a}_\phi\\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ F_\rho & \rho F_\theta & \rho\sin\theta F_\phi\end{vmatrix}$$

(Spherical coordinates) .....(16)

Ques 17) Find the gradient of the following scalar fields:

1) $\mathbf{F = x^2 y + e^z}$
2) $\mathbf{V = rz\sin\phi + z^2\cos^2\phi + r^2}$
3) $\mathbf{S = \cos\theta\sin\phi\,\ln\rho + \rho^2\phi}$.

Ans:

1) The gradient in Cartesian coordinates is given as,

$$\nabla F=\frac{\partial F}{\partial x}\hat{a}_x+\frac{\partial F}{\partial y}\hat{a}_y+\frac{\partial F}{\partial z}\hat{a}_z=2x\hat{a}_x+x^2\hat{a}_y+e^z\hat{a}_z$$

2) The gradient in cylindrical coordinates is given as,

$$\nabla V=\frac{\partial V}{\partial r}\hat{a}_r+\frac{1\partial V}{r\partial\phi}\hat{a}_\phi+\frac{\partial V}{\partial z}\hat{a}_z$$

$$=(z\sin\phi+2r)\hat{a}_r+\frac{1}{r}(rz\cos\phi-z^2 2\cos\phi\sin\phi)\hat{a}_\phi$$

$$+(r\sin\phi+2z\cos^2\phi)\hat{a}_z$$

$$=(z\sin\phi+2r)\hat{a}_r+\left(z\cos\phi-\frac{z^2}{r}\sin 2\phi\right)\hat{a}_\phi$$

$$+(r\sin\phi+2z\cos^2\phi)\hat{a}_z$$

3) The gradient in spherical coordinates is given as,

$$\nabla S=\frac{\partial S}{\partial\rho}\hat{a}_\rho+\frac{1}{\rho}\frac{\partial S}{\partial\theta}\hat{a}_\theta+\frac{1}{\rho\sin\theta}\frac{\partial S}{\partial\phi}\hat{a}_\phi$$

$$=\left(\frac{\cos\theta\sin\phi}{\rho}+2\rho\phi\right)\hat{a}_\rho+\frac{1}{\rho}\sin\theta\sin\phi\ln\rho\hat{a}_\theta$$

$$+\frac{1}{\rho\sin\theta}(\cos\theta\cos\phi\ln\rho+\rho^2)\hat{a}_\phi$$

$$=\left(\frac{\cos\theta\sin\phi}{\rho}+2\rho\phi\right)\hat{a}_\rho+\frac{\sin\theta\sin\phi}{\rho}\ln\rho\hat{a}_\theta$$

$$+\left(\frac{\cot\theta}{\rho}\cos\phi\ln\rho+\rho\,\text{cosec}\,\theta\right)\hat{a}_\phi$$

Ques 18) Find the rate at which the scalar function $\mathbf{V = r^2\sin 2\phi}$ in cylindrical coordinates increases in the direction of the vector $\mathbf{\bar{A}=\bar{a}_r+\bar{a}_\phi}$ at the point $\left(2,\frac{\pi}{4},0\right)$.

Or

Find the gradient of the scalar function $\mathbf{V = r^2\sin 2\phi}$ and the directional derivative of the function in the direction $(\bar{a}_r+\bar{a}_\varphi)$ at the point $\left(2,\frac{\pi}{4},0\right)$.

Ans: The gradient in cylindrical coordinates is given as,

$$\nabla V=\frac{\partial V}{\partial r}\hat{a}_r+\frac{1}{r}\frac{\partial V}{\partial\phi}\hat{a}_\phi+\frac{\partial V}{\partial z}\hat{a}_z=2r\sin 2\phi\hat{a}_r+2r\cos 2\phi\hat{a}_\phi$$

$$=2r(\sin 2\phi\hat{a}_r+\cos 2\phi\bar{a}_\phi)$$

The direction derivative is given as,

$$\nabla V\cdot\hat{a}_A=\nabla V\cdot\frac{\vec{A}}{|\vec{A}|}$$

$$=(2r\sin 2\phi\hat{a}_r+2r\cos 2\phi\hat{a}_\phi)\cdot\left(\frac{\hat{a}_r+\hat{a}_\phi}{\sqrt{2}}\right)$$

$$=\sqrt{2}r\sin 2\phi+\sqrt{2}r\cos 2\phi$$

At $\left(2,\frac{\pi}{4},0\right)$, the directional derivative is given as,

$$\nabla V\cdot\hat{a}_A=\sqrt{2}\times 2\sin\frac{\pi}{2}+\sqrt{2}\times 2\cos\frac{\pi}{2}=2\sqrt{2}$$

Ques 19) Determine the divergence of the vector field given as,

$$\vec{V}=\rho\cos\theta\hat{a}_\rho-\frac{1}{\rho}\sin\theta\hat{a}_\theta+2\rho^2\sin\theta\hat{a}_\varphi$$

Ans: The divergence in spherical coordinate system is given as,

$$\nabla\cdot\vec{V}=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2V_\rho)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta)+\frac{1}{\rho\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^3\cos\theta)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{\rho}\sin^2\theta\right)+\frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}(2\rho^2\sin\theta)$$

$$=3\cos\theta-\frac{1}{\rho^2\sin\theta}2\sin\theta\cos\theta+0=\left(3-\frac{2}{\rho^2}\right)\cos\theta$$

Ques 20) Determine the curl of the following vector fields:

1) $\vec{F}=x^2y\hat{a}_x+y^2z\hat{a}_y-2xz\hat{a}_z$

2) $\vec{A}=r^2\sin\phi\hat{a}_r+r\cos^2\phi\hat{a}_\phi+z\tan\phi\hat{a}_z$

3) $\vec{V}=\frac{\sin\phi}{\rho^2}\hat{a}_\rho-\frac{\cos\phi}{\rho^2}\hat{a}_\phi$

Ans:

1) $\vec{F}=x^2y\hat{a}_x+y^2z\hat{a}_y-2xz\hat{a}_z$

The curl in Cartesian coordinate system is given as,

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & y^2z & -2xz\end{vmatrix}$$

$$=-y^2\hat{a}_x+2z\hat{a}_y-x^2\hat{a}_z$$

2) $\vec{A}=r^2\sin\phi\hat{a}_r+r\cos^2\phi\hat{a}_\phi+z\tan\phi\hat{a}_z$

The curl in cylindrical coordinate system is given as,

$$\nabla\times\vec{A}=\left(\frac{1}{r}\right)\begin{vmatrix}\hat{a}_r & r\hat{a}_\phi & \hat{a}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$$

$$=\left[\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right]\hat{a}_r+\left[\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right]\hat{a}_\phi+\left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right]\hat{a}_z$$

$$=\frac{1}{r}(z\sec^2\phi-0)\hat{a}_r+(0-0)\hat{a}_\phi+\left(\frac{1}{r}\right)(2r\cos^2\phi-r^2\cos\phi)\hat{a}_z$$

$$=\frac{1}{r}[z\sec^2\phi\hat{a}_r+(2r\cos^2\phi-r^2\cos\phi)\hat{a}_z]$$

3) $\vec{V}=\frac{\sin\phi}{\rho^2}\hat{a}_\rho-\frac{\cos\phi}{\rho^2}\hat{a}_\phi$

The curl in spherical coordinate system is given as,

$$\nabla\times\vec{V}=\left(\frac{1}{\rho^2\sin\theta}\right)\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\theta & \rho\sin\theta\hat{a}_\phi\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ V_\rho & \rho V_\theta & \rho\sin\theta V_\phi\end{vmatrix}$$

$$=\left(\frac{1}{\rho\sin\theta}\right)\left[\frac{\partial}{\partial\theta}(\sin\theta V_\phi)-\frac{\partial V_\theta}{\partial\phi}\right]\hat{a}_\rho+\frac{1}{\rho}\left[\frac{1}{\sin\theta}\frac{\partial V_\rho}{\partial\rho}-\frac{\partial}{\partial\rho}(\rho V_\phi)\right]\hat{a}_\theta+\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho V_\theta)-\frac{\partial V_\rho}{\partial\theta}\right]\hat{a}_\phi$$

$$=\left(\frac{1}{\rho\sin\theta}\right)\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\cos\phi}{\rho^2}\right)-0\right]\hat{a}_\rho+\frac{1}{\rho}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{\rho^2}\right)-\frac{\partial}{\partial\rho}\left(\rho\frac{\cos\phi}{\rho^2}\right)\right]\hat{a}_\theta+\frac{1}{\rho}\left[0-\frac{\partial}{\partial\theta}\left(\frac{\sin\phi}{\rho^2}\right)\right]\hat{a}_\phi$$

$$=\left(\frac{1}{\rho\sin\theta}\right)\left(\frac{\cos\theta\cos\phi}{\rho^2}\right)\hat{a}_\rho+\frac{1}{\rho}\left(\frac{\cos\phi}{\rho^2\sin\theta}+\frac{\cos\phi}{\rho^2}\right)\hat{a}_\theta$$

$$\therefore\ \nabla\times\vec{V}=\left(\frac{1}{\rho^3}\cot\theta\cot\phi\right)\hat{a}_\rho+\frac{1}{\rho^3}\left(\frac{\cos\phi}{\sin\phi}+\cos\phi\right)\hat{a}_\theta$$

Ques 21) If $\vec{A}=xz^3\hat{i}-2x^2yz\hat{j}+2yz^4\hat{k}$, find Curl $\vec{A}$ at the point (1, –1, 1).

Ans:

$$\nabla\times\vec{A}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xz^3 & -2x^2yz & 2yz^4\end{vmatrix}=(2z^4+2x^2y)\hat{a}_x+(3xz^2-0)\hat{a}_y+(-4xyz-0)\hat{a}_z$$

$$=(2z^4+2x^2y)\hat{a}_x+3xz^2\hat{a}_y-4xyz\hat{a}_z)$$

$$\therefore\ \nabla.\vec{A}\Big|_{(1,-1,1)}=(2-2)\hat{a}_x+3\hat{a}_y+4\hat{a}_z=3\hat{a}_y+4\hat{a}_z$$

Ques 22) State and explain divergence theorem.

Ans: Divergence theorem

Let $\vec{F}$ be a smooth vector field defined on a solid region V with boundary surface A oriented outward. This is showed that:

$$\int_A\vec{F}\cdot d\vec{A}=\int_V \text{div}\,\vec{F}dV$$

For the Divergence Theorem, use the same approach as used for Green's Theorem; first prove the theorem for rectangular regions, then use the change of variables formula to prove it for regions parameterised by rectangular regions, and finally paste such regions together to form general regions.

Consider a smooth vector field $\vec{F}$ defined on the rectangular solid V $a \le x \le b, c \le y \le d, e \le z \le f$. We start by computing the flux of $\vec{F}$ through the two faces of V perpendicular to the x-axis, A_1 and A_2, both oriented outward.

$$\int_{A_1} \vec{F}\cdot d\vec{A} + \int_{A_2} \vec{F}\cdot d\vec{A} = -\int_e^f \int_c^d F_1(a,y,z)\,dy\,dz + \int_e^f \int_c^d F_1(b,y,z)\,dy\,dz$$

$$= \int_e^f \int_c^d (F_1(b,y,z) - F_1(a,y,z))\,dy\,dz$$

By the fundamental theorem of calculus:

$$F_1(b,y,z) - F_1(a,y,z) = \int_a^b \frac{\partial F_1}{\partial x} dx$$

So,

$$\int_{A_1} \vec{F}\cdot d\vec{A} + \int_{A_2} \vec{F}\cdot d\vec{A} = \int_e^f \int_c^d \int_a^b \frac{\partial F_1}{\partial x} dx\,dy\,dz = \int_V \frac{\partial F_1}{\partial x} dV$$

By a similar argument, we can show:

$$\int_{A_3} \vec{F}\cdot d\vec{A} + \int_{A_4} \vec{F}\cdot d\vec{A} = \int_V \frac{\partial F_2}{\partial y} dV \quad \text{and}$$

$$\int_{A_5} \vec{F}\cdot d\vec{A} + \int_{A_6} \vec{F}\cdot d\vec{A} = \int_V \frac{\partial F_3}{\partial z} dV$$

Adding these, we get:

$$\int_A \vec{F}\cdot d\vec{A} = \int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \int_V \operatorname{div} \vec{F}\, dV$$

This is the Divergence Theorem for the region V.

Ques 23) Give that $\vec{A} = 30e^{-r}\,\vec{a}_r - 2z\,\vec{a}_z$ in the cylindrical co-ordinates. Evaluate both sides of the divergence theorem for the volume enclosed by r = 2, z = 0 and z = 5.

Ans: The divergence theorem states that;

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V (\nabla\cdot\vec{A})dv$$

$$\text{Now, } \oint_S \vec{A}\cdot d\vec{S} = \left[\oint_{side} + \oint_{top} + \oint_{bottom}\right] \vec{A}\cdot d\vec{S}$$

Consider $d\vec{S}$ normal to $\vec{a}_r$ direction which is for the side surface.

$$\therefore\ d\vec{S} = r\,d\phi\,dz\,\vec{a}_r$$

$$\therefore\ \vec{A}\cdot d\vec{S} = (30e^{-r}\,\vec{a}_r - 2z\,\vec{a}_z)\cdot r\,d\phi\,dz\,\vec{a}_r$$

$$= 30re^{-r}(\vec{a}_r\cdot\vec{a}_r)d\phi\,dz = 30re^{-r}d\phi\,dz$$

$$\therefore\ \oint_{side} \vec{A}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{5} 30re^{-r}d\phi\,dz \quad \text{with } r = 2$$

$$= 30\times 2\times e^{-2}\times[\phi]_0^{2\pi}\times[z]_0^5 = 255.1$$

The $d\vec{S}$ on top has direction $\vec{a}_z$ hence for top surface,

$$d\vec{S} = r\,dr\,d\phi\,\vec{a}_r$$

$$\therefore\ \vec{A}\cdot d\vec{S} = (30e^{-r}\bar{a}_r - 2z\,\vec{a}_z)\cdot r\,dr\,d\phi\,\vec{a}_r$$

$$= -2\,z\,r\,dr\,d\phi \qquad (\bar{a}_z\cdot\bar{a}_z = 1)$$

$$\therefore\ \oint_{top} \vec{A}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{r=0}^{2} -2zr\,dr\,d\phi \quad \text{with } z = 5$$

$$= -2\times 5\times\left[\frac{r^2}{2}\right]_0^2 \times[\phi]_0^{2\pi} = -40\pi$$

While $d\bar{S}$ for bottom has direction $-\vec{a}_z$ hence for bottom surface,

$$d\vec{S} = r\,dr\,d\phi(-\vec{a}_r)$$

$$\therefore\ \vec{A}\cdot d\vec{S} = (30e^{-r}\,\vec{a}_r - 2z\,\vec{a}_z)\cdot r\,dr\,d\phi(-\vec{a}_r)$$

$$= 2\,z\,r\,dr\,d\phi \qquad (\bar{a}_z\cdot\bar{a}_z = 1)$$

But z = 0 for the bottom surface, as shown in the **figure 1.10**.

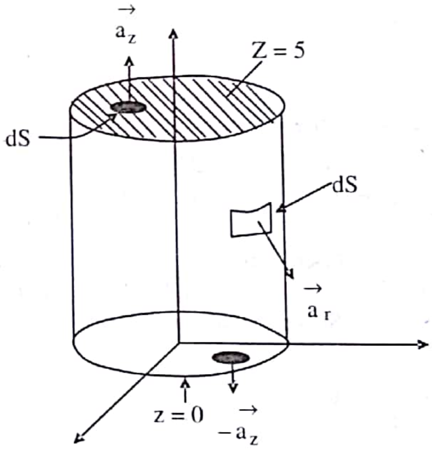

Figure 1.10:

$$\therefore\ \oint_S \vec{A}\cdot d\vec{S} = 255.1 - 40\pi + 0$$

$$= 129.4363$$

This is the left hand side of divergence theorem;

Now evaluate $\int_V (\nabla\cdot\vec{A})dv$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_\phi}{\partial z}$$

And, $A_r = 30\,e^{-r}$, $A_\phi = 0$, $A_z = -2z$

$$\therefore\ \nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(30re^{-r}) + 0 + \frac{\partial}{\partial z}(-2z)$$

$$= \frac{1}{r}\{30r(-e^{-r}) + 30e^{-r}(1)\} + (-2)$$

$$= -30e^{-r} + \frac{30}{r}e^{-r} - 2$$

$$\therefore \int_v (\nabla.\bar{A})dv = \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=0}^{2}\left(-30e^{-r}+\frac{30}{r}e^{-r}-2\right)r\,dr\,d\phi\,dz$$

$$= \int_{z=0}^{5}\int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(-30re^{-r}+30e^{-r}-2r\right)dr\,d\phi\,dz$$

$$= \left\{-30r\left[\frac{e^{-r}}{-1}\right]-\int(-30)\left[\frac{e^{-r}}{-1}\right]dr+30\left[\frac{e^{-r}}{-1}\right]-\left[2\frac{r^2}{2}\right]\right\}[z]_0^5[\phi]_0^{2\pi}$$

Obtained using integration by parts:

$$=\left[30re^{-r}+30e^{-r}-30e^{-r}-r^2\right]_0^2[5][2\pi]$$

$$= [60e^{-2}-2^2][10\pi] = 129.437$$

This is same as obtained from the left hand side.

Ques 24) State and prove stokes' theorem.

Ans: Stokes' Theorem

This theorem states that the line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over the surface bounded by the path.

Mathematically can be written as,

$$\oint_L \bar{F}.d\bar{\ell} = \iint_S (\nabla\times\bar{F}).d\bar{S} = \iint_S (\nabla\times\bar{F}).\hat{a}_n dS \quad \text{..... (1)}$$

Where S is the surface enclosed by the path L. The positive direction of $d\bar{S}$ is related to the positive sense of defining L according to the right-hand rule.

Proof of Stokes' Theorem

Consider an oriented surfaces A, bounded by the curve B. We want to prove Stokes' Theorem:

$$\int_A \text{curl}\,\bar{F}\cdot d\bar{A} = \int_B \bar{F}\cdot d\bar{r}$$

We suppose that A has a smooth parameterisation $\bar{r} = \bar{r}\,(s,t)$, so that A corresponds to a region R in the st-plane, and B corresponds to the boundary C of R as shown in **figure 1.11**. We prove Stokes' theorem for the surface A and a continuously differentiable vector field $\bar{F}$ by expressing the integrals on both sides of theorem in terms of s and t, and using Green's Theorem in the st-plane.

First, we convert the line integral $\int_B \bar{F}\cdot d\bar{r}$ into a line integral around C:

$$\int_B \bar{F}.d\bar{r} = \int_C \bar{F}\cdot\frac{\partial\bar{r}}{\partial s}ds + \bar{F}\cdot\frac{\partial\bar{r}}{\partial t}dt$$

So if we define a 2-dimensional vector field $\bar{G}=(G_1, G_2)$ on the st-plane by,

$$G_1 = \bar{F}.\frac{\partial\bar{r}}{\partial s} \text{ and } G_2 = \bar{F}.\frac{\partial\bar{r}}{\partial t}$$

then, $\int_B \bar{F}\cdot d\bar{r} = \int_C \bar{G}\cdot d\bar{s}$

using $\bar{s}$ to denote the position vector of a point in the st-plane.

What about the flux integral $\int_A \text{curl}\ \bar{F}\cdot d\bar{A}$ that occurs on the other side of Stokes' Theorem?

In terms of the parameterisation,

$$\int_A \text{curl}\ \bar{F}\cdot d\bar{A} = \int_R \text{curl}\,\bar{F}.\frac{\partial\bar{r}}{\partial s}\times\frac{\partial\bar{r}}{\partial t}dsdt$$

Since,

$$\text{curl}\,\bar{F}.\frac{\partial\bar{r}}{\partial s}\times\frac{\partial\bar{r}}{\partial t} = \frac{\partial G_2}{\partial s}-\frac{\partial G_1}{\partial t}$$

$$\text{Hence, } \int_A \text{curl}\ \bar{F}\cdot d\bar{A} = \int_R\left(\frac{\partial G_2}{\partial s}-\frac{\partial G_1}{\partial t}\right)dsdt$$

We have already seen that;

$$\int_B \bar{F}\cdot d\bar{r} = \int_C \bar{G}\cdot d\bar{s}$$

By Green's Theorem, the right-hand sides of the two equations are equal. Hence the left-hand sides are equal as well, which is what we had to prove for Stokes' Theorem.

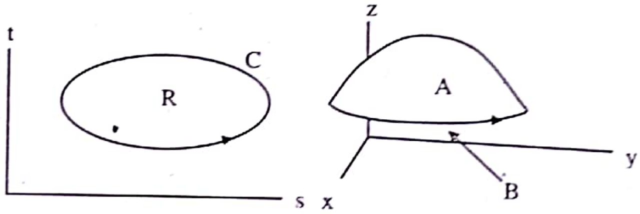

Figure 1.11: Region R in the st-plane and the Corresponding Surface A in xyz-Space; the Curve C Corresponds to the Boundary of B

Ques 25) Given $\bar{A} = 2r\cos\phi\bar{a}_r + r\bar{a}_\phi$ in cylindrical coordinates. For the path shown in the Figure 1.12, verify Stoke's theorem.

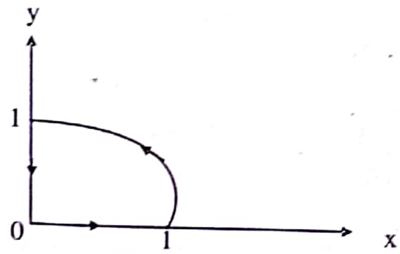

Figure 1.12

Ans: According to Stoke's theorem,

$$\oint_L \bar{A}\cdot d\bar{L} = \oint_S \nabla(\times\bar{A})\cdot d\bar{S}$$

For L.H.S., $d\bar{L} = dr\bar{a}_r + rd\phi\bar{a}_\phi + dz\bar{a}_z$ in cylindrical system.

There are three paths for which evaluate L.H.S.

$$\therefore \oint_L \bar{A}\cdot d\bar{L} = \int_{path1} A_r dr + \int_{path2} A_r dr + \int_{path3} A_\phi[r\,d\phi]$$

Where $A_r = 2r\cos\phi$, $A_\phi = r$, $A_z = 0$

For path 1, the direction is $\bar{a}_r$ while for path 2 also the direction is $\bar{a}_r$. For path 3 the direction is $\bar{a}_\phi$.

$$\therefore \oint_L \bar{A}\cdot d\bar{L} = \int_{r=1}^{0} 2r\cos\phi\, dr + \int_{r=0}^{1} 2r\cos\phi\, dr \int_{\phi=0}^{\pi/2} r^2 d\phi$$

$\phi = 90°$ $\quad \phi = 0°$ $\quad r = 1$

$$= 0 + 2\cos 0°\left[\frac{r^2}{2}\right]_0^1 + (1)^2[\phi]_0^{\pi/2}$$

$$= 2\times\frac{1}{2}+\frac{\pi}{2} = \frac{2+\pi}{2} = 2.5707A \quad \text{....L.H.S.}$$

For R.H.S. find $\nabla\times\bar{A}$ first in cylindrical system.

$$\therefore \nabla\times\bar{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{dA_\phi}{\partial z}\right]\bar{a}r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi$$

$$+\left[\frac{1}{4r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right]\bar{a}_z$$

$$= 0 + 0 + \left[\frac{1}{r}\frac{\partial r^2}{\partial r} - \frac{1}{r}\frac{\partial[2r\cos\phi]}{\partial\phi}\right]$$

$$= [2 - 2(-\sin\phi)]\bar{a}_z = [2 + 2\sin\phi]\bar{a}_z$$

As surface is in $\bar{a}_z$ direction hence

$$d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$$

$$\therefore \left[\nabla\times\bar{A}\right]\cdot d\bar{S} = [2+2\sin\phi]r\, dr\, d\phi$$

$$\therefore \text{R.H.S.} = \oint_s(\nabla\times\bar{A})\cdot d\bar{S} = \int_{\phi=0}^{\pi/2}\int_{r=0}^{1}[2+2\sin\phi]r\, dr\, d\phi$$

$$= \left[\frac{r^2}{2}\right]_0^1 \times [2\phi - 2\cos\phi]_0^{\pi/2}$$

$$= \frac{1}{2}\times\left[\frac{2\times\pi}{2} - 0 - 0 + 2\right] = \frac{\pi+2}{2} = 2.5707A$$

Thus as L.H.S. = R.H.S., the Stoke's theorem is verified.

Module 2

Static Electric Field

ELECTRIC FIELDS

Ques 1) State and explain Coulomb's law.

Ans: Coulomb's Law
This law states that the force between two point charges:
1) Acts along the line joining the two charges.
2) Is directly proportional to the product of the two charges.
3) Is inversely proportional to the square of the distance between the charges.

Let us consider two point charges Q_1 and Q_2 with separation distance R, as shown in **figure 2.1 (a)** and **figure 2.1 (b)**:

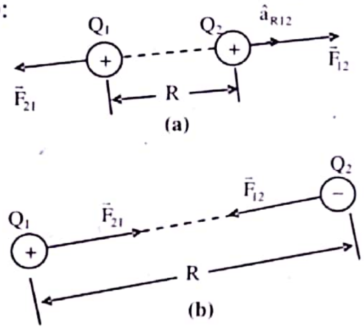

Figure 2.1: Coulomb Interaction between Two Point Charges (a) Like Charges, and (b) Unlike Charges

The force exerted by Q_1 on Q_2 is,

$$\vec{F}_{12} \propto \frac{Q_1 Q_2}{R^2}\hat{a}_{R12}$$

$$\Rightarrow \quad \vec{F}_{12} = k\frac{Q_1 Q_2}{R^2}\hat{a}_{R12} \quad .....(1)$$

Where, k is the proportionality constant, which takes into account the effect of the medium in which the charges are placed and $\hat{a}_{R12}$ is a unit vector directed from Q_1 to Q_2.

In SI unit, charges expressed in Coulomb (C), the distance expressed in metre (m) and the force expressed in Newton (N), the proportionality constant is given as,

$$k = \frac{1}{4\pi\varepsilon}$$

Where,
ε = Permittivity of the medium = $\varepsilon_0\varepsilon_r$

ε_0 = Permittivity of free space

$$= \frac{1}{36\pi \times 10^9} = 8.854 \times 10^{-12} F/m$$

ε_r = Relative permittivity of the medium

Thus from equation (1), the Coulomb's law in SI unit becomes,

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}\hat{a}_{R12} \quad .....(2)$$

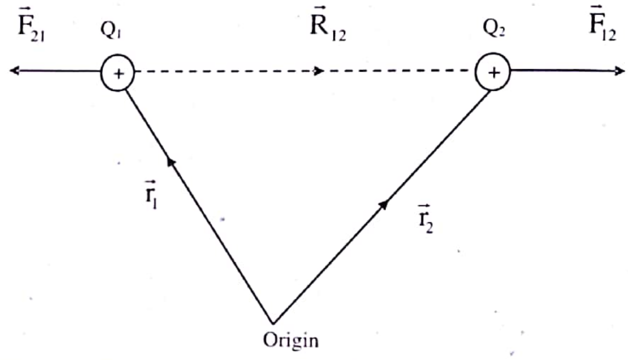

Figure 2.2: Coulomb Vector Force between Two Point Charges

Similarly, force exerted by Q_2 on Q_1 is,

$$\vec{F}_{21} \propto \frac{Q_1 Q_2}{R^2}\hat{a}_{R21}$$

$$\Rightarrow \quad \vec{F}_{21} = k\frac{Q_1 Q_2}{R^2}\hat{a}_{R21} = -\vec{F}_{12}$$

Where, two charges have the position vectors of $\vec{r}_1$ and $\vec{r}_2$ respectively, as shown in **figure 2.2**.

Ques 2) Discuss the applications of coulomb's law?

Ans: Applications of Coulomb's law
Coulomb's law is used to:
1) Find the force between a pair of charges.
2) Find the potential at a point due to a fixed charge.
3) Find the electric field at a point due to a fixed charge.
4) Find the displacement flux density indirectly.
5) Find the potential and electric field due to any type of charge distribution.
6) Find the charge if the force and the electric field are known.

Ques 3) Define Electric field and electric field intensity and formulate it for various charge distributions/charge densities.

Ans: Electric Field

For an electric charge, there is a region in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region, is called the **electric field** of that charge.

Electric Field Intensity

Electric field intensity $\vec{E}$ is the force per unit charge when placed in the field.

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q}$$

Or Simply $\vec{E} = \dfrac{\vec{F}}{Q}$ .....(1)

It is seen that the field intensity is in the same direction as the force and is expressed in Newton per Coulomb (N/C) and volt per metre (V/m).

Electric field is also defined as a negative gradient of a potential due to a charge, that is,

$E \equiv -\nabla.V$, volts/metre

Thus, if a point charge Q_1 is present at position vector $\vec{r}_1$, then the field intensity due to the charge Q at position vector $\vec{r}$ is,

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon R^2}\hat{a}_R = \frac{Q_1(\vec{r}_1 - \vec{r})}{4\pi\varepsilon|\vec{r}_1 - \vec{r}|^3}$$

Electric field due to various charge distribution

By definition, line charge density is given by

$$\rho_L = \frac{dQ}{dL} \text{ C/m}$$

Surface charge density is given by

$$\rho_S = \frac{dQ}{dS} \text{ C/m}^2$$

Volume charge density is defined as,

$$\rho_v = \underset{\Delta v \to 0}{\text{Lt}} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv}$$

Electric field due to surface charge density,

$$E = \frac{\rho_S}{2\varepsilon_0} a_n$$

Electric field due to a uniform infinite line charge is given by,

$$E = \frac{\rho_L}{2\pi\varepsilon_0 r} a_\rho$$

Ques 4) Find the electric field intensity due to infinite line charged wire (line charged).

Ans: Electric Field due to a Line Charge

Let us consider an infinitely long charged wire of negligible thickness and having a constant linear charge density ρ. Let a point P be at a distance y from the wire, as shown in **figure 2.3**.

It is required to find the electric field intensity at P. Let us assume that the wire is made up of a number of infinitely small elements of length, dx. Let one of such elements be at a distance x, as shown in **figure 2.3.** Let the small charge on element be dq.

$$dq = \rho\, dx$$

The field due to this charge dq at point is:

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dq}{(NP)^2} = \frac{dq}{4\pi\varepsilon_0 r^2}$$

The x and y components of dE are:

$dE_x = -dE \sin\theta$ and $dE_y = -dE\cos\theta$

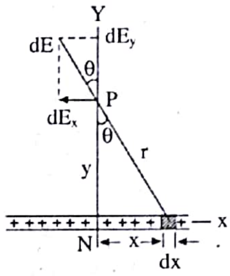

Figure 2.3

The x-components of field at P cancel out each other's effect. Therefore, the net field will be due to y-components only and is directed along y-axis.

The resultant field is,

$$E = \int_{x=-\infty}^{x=+\infty} dE_y = \int_{x=-\infty}^{x=+\infty} dE\cos\theta = \int_0^{\infty} 2dE\cos\theta = \int_0^{\infty} \frac{2dq}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$= \int_0^{\infty} \frac{2\rho dx}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{\rho}{2\pi\varepsilon_0}\int_0^{\infty} \frac{dx}{r^2}\cos\theta$$

From **figure 2.3**, we have; $\dfrac{x}{y} = \tan\theta$

$$\therefore \frac{dx}{y} = \sec^2\theta \quad \text{or} \quad dx = y\sec^2\theta\, d\theta,$$

Also, $x^2 + y^2 = r^2$

∴

$$E = \frac{\rho}{2\pi\varepsilon_0}\int_0^{\theta=\pi/2} \frac{y\sec^2\theta d\theta}{x^2 + y^2}\cos\theta = \frac{\rho}{2\pi\varepsilon_0}\int_0^{\theta=\pi/2} \frac{y\sec^2\theta d\theta}{y^2\tan^2\theta + y^2}\cos\theta$$

$$= \frac{\rho}{2\pi\varepsilon_0}\int_0^{\theta=\pi/2} \frac{y\sec^2\theta d\theta}{y^2\sec^2\theta}\cos\theta = \frac{\rho}{2\pi\varepsilon_0}\int_0^{\theta=\pi/2} \frac{\cos\theta}{y} d\theta$$

$$= \frac{\rho}{2\pi\varepsilon_0 y}[\sin\theta]_0^{\pi/2}, \quad E = \frac{\rho}{2\pi\varepsilon_0 y}$$

Ques 5) Find expression for electric field intensity for an infinite sheet charge having line charge density ρ_s C/m².

Or

Find expression for electric field intensity for an infinite sheet charge.

Ans: Electric Field Intensity For An Infinite Sheet Charge

Consider an infinite sheet of charge in the y-z plane as shown in **figure 2.4** having a uniform charge of ρ_s C/m².

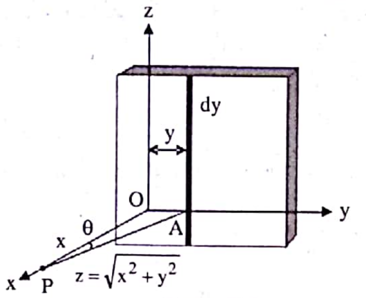

Figure 2.4

For easy analysis, divide the sheet into differential width strips (dy).

$$\rho_L = \rho_s\, dy$$

The distance between the point P the line charge,

$$Z^2 = X^2 + Y^2$$

Since the sheet lies in y-z plane the field components due to y and z will be cancelled out at the point at which the field is to be determined.

Only E_x component is present and hence the field is a function of x alone for an infinite sheet of charge on y-z plane. Hence the electric field intensity due to an infinite line charge is given as:

$$E = \frac{\rho_L}{2\pi\varepsilon_0 \rho} a_\rho$$

Magnitude E_x at point P due to an infinite differential width strip (dy).

From the triangle OAP,

$$dE_x = \frac{\rho_s dy}{2\pi\varepsilon_0 z}\cos\theta; \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

Hence E_x is given as:

$$dE_x = \frac{\rho_s dy}{2\pi\varepsilon_0 z} \times \frac{x}{\sqrt{x^2+y^2}}$$

From $-\infty$ to $+\infty$,

$$E_x = \int_{-\infty}^{+\infty} dE_x = \frac{\rho_s}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{x\,dy}{x^2+y^2}$$

$$= \frac{\rho_s}{2\pi\varepsilon_0}\tan^{-1}\frac{y}{x}\Big|_{-\infty}^{+\infty} \Rightarrow E_x = \frac{\rho_s}{2\varepsilon_0}$$

If the field intensity is obtained at point P on the negative axis, then E will be,

$$E_x = -\frac{\rho_s}{2\varepsilon_0}; \qquad x < 0$$

In general, electric field intensity for an infinite sheet of charge is given as,

$$E = \frac{\rho_s}{2\varepsilon_0}\hat{a}_n$$

Where $\hat{a}_n$ is a unit vector normal to the sheet.

The electric field intensity (E) points away from the plane if ρ_s is positive and towards the plane if ρ_s is negative. The magnitude of the electric field is a constant; the magnitude is independent of the distance from the infinite plane.

This is because no matter how far the point is from the infinite sheet, the distance becomes incomparable with the dimensions of the plane. Hence it seems the point is very close to the infinite plane.

In a parallel plate capacitor the electric field intensity between the two plates having equal and opposite charge is given by:

$$E = \frac{\rho_s}{2\varepsilon}\hat{a}_n + \frac{-\rho_s}{2\varepsilon}(-\hat{a}_n) = \frac{\rho_s}{\varepsilon}\hat{a}_n$$

The first –ve sign denotes –ve charges on one plate and the second – ve sign denotes opposite direction.

Ques 6) Find expression for electric field intensity for continuous volume charge distribution.

Ans: Field Intensity of a Volume Charge Distribution

Continuous distribution of charge in a region of volume is specified by charge per unit volume which is called volume charge density. Its unit is coulomb per metre³ (C/m³), and it is generally denoted by the symbol ρ.

If the charge distribution is not uniform, the volume charge density,

$$\rho = \lim_{\Delta v \to 0}\frac{\Delta Q}{\Delta v}$$

Where, ΔQ is charge contained in a volume element Δv.

Referring to **figure 2.5** let dv′ be a differential volume centre at P′(x′,y′,z′) within the charge distribution and P (x,y,z) is the point at which the field intensity is required.

If charge density at P′ is $\rho(x',y',z')$, charge contained in the differential volume,

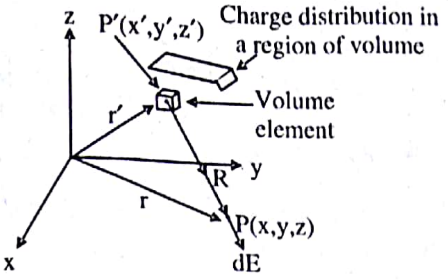

Figure 2.5: Field at P due to Charge in a Differential Volume Element

$dQ = \rho(x', y', z')dv'$

The displacement of P from P′ is:

$R = r = r' = (x - x')\,u_x + (y - y')\,u_y + (z - z')\,u_z$

Thus, the field intensity at P due to the differential charge,

$$dE = \frac{\rho(x', y', z')dv'}{4\pi R^3}R \qquad ....(1)$$

The total field intensity is found by integrating equation (1) over the region of charge.

$$E = \int_v \frac{\rho(x', y', z')R}{4\pi R^3}dv'$$

It may be noted that is not easy to evaluate the integral even for uniform distribution of charge as it is a triple integral and the integrand is a vector function.

Ques 7) Two large sheets of charge with densities ρ_{S1} and $-\rho_{S2}$ are located at x = 0 and x = a. Find field intensity in all the regions.

Ans: The directions of field due to the charge distributions are as shown if **figure 2.6.**

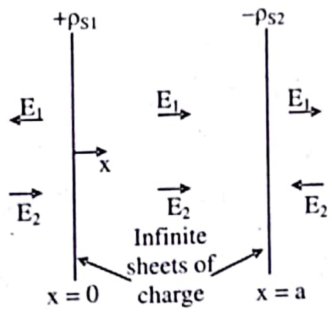

Figure 2.6: Showing Directions of Field Intensities Produced by Two Parallel Sheets of Charge

1) In the region x < 0

$$E = E_2 - E_1 = \frac{\rho_{s2} - \rho_{s1}}{2\varepsilon}u_x \text{ V/m}$$

2) In the region 0 < x < a

$$E = E_2 + E_1 = \frac{\rho_{s2} + \rho_{s1}}{2\varepsilon}u_x \text{ V/m}$$

3) In the region x > a

$$E = E_2 - E_1 = \frac{\rho_{s1} - \rho_{s2}}{2\varepsilon}u_x \text{ V/m}$$

Ques 8) Explain the following:
1) **Electric Flux**
2) **Electric Flux Density**

Ans: Electric Flux

Electric flux is defined as the number of lines of force that pass through a surface placed in the vector field.

Electric flux may also be described mathematically, as the product of the surface area $\vec{ds}$ and the components of the electric field $\vec{E}$ normal to the surface.

i.e., $\phi = \vec{E} \cdot \vec{ds}$

A unit charge is supposed to emanate one flux. Thus, in case of an isolated charge q coulomb, the flux is (ϕ = q), and is independent of the nature of the medium. In such a case, the unit of flux is that of charge, i.e., coulomb.

The relation between $\vec{D}$ and $\vec{E}$ is, $\vec{D} = \varepsilon\vec{E}$ or $\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$. Here, the flux density at any point in an electric field is $\varepsilon_0\varepsilon_r$ times the electric field at the point.

Electric Flux Density $\vec{D}$

Electric flux density as the flux passing per unit area of a section, held normal to the direction of the flux.

Thus, if ϕ flux passes normally through an area Am^2, the flux density is:

$$\vec{D} = \frac{\text{flux}}{\text{Area}} = \frac{\phi}{A}Cm^2$$

Electric flux density is a vector quantity having both magnitude and direction. The direction is that of the electric lines of force or flux.

Ques 9) Two parallel rectangular plates measuring 20cm by 40cm carry an electric charge of 0.2μC. Calculate the electric flux density. If the plates are spaced 5mm apart and the voltage between them is 0.25kV, determine the electric field strength.

Ans: Given that

Area = 20cm × 40cm = $800cm^2$ = 800 × $100^{-4}m^2$ and charge Q = 0.2μC = 0.2 × 10^{-6}C

Electric Flux Density

$$D = \frac{Q}{A} = \frac{0.2\times10^{-6}}{800\times10^{-4}} = \frac{0.2\times10^{4}}{800\times10^{6}}$$

$$= \frac{2000}{800}\times10^{-6} = 2.5\mu C/m^2$$

Voltage V = 0.25kV = 250V and plate spacing, d = 5mm = 5 × 10^{-3}m.

Electric Field Strength

$$E = \frac{V}{d} = \frac{250}{5\times10^{-3}} = 50kV/m$$

Ques 10) A point charge of 2.0 nC is at the centre of a sphere, as shown in figure 2.7. Find the flux crossing a part of the sphere defined by $30^0 \le \phi \le 60^0$ and $0 \le \phi \le 90^0$

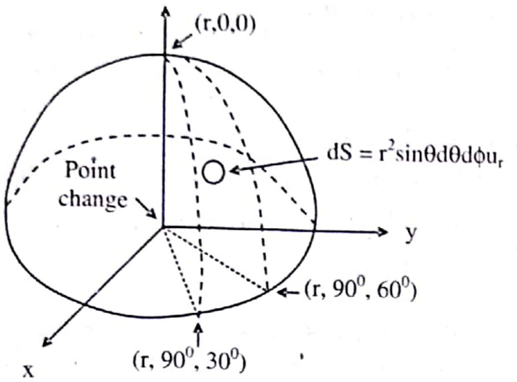

Figure 2.7

Ans: Flux density on the surface of a sphere of radius r,

$$D = \frac{2\times10^{-9}}{4\pi r^2} u_r \, C$$

A differential area in r direction,

$dS = r^2 \sin\theta d\theta d\phi u_r \, m^2$

Flux through the surface is given by,

$$\psi = \int_s D\cdot dS = \frac{10^{-9}}{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \, d\phi = \frac{1}{12} nC$$

Ques 11) The linear charge density of an infinite line charge located along the axis of a cylinder of radius r is 8 nC/m. The axis of the cylinder is the z axis of cylindrical coordinates. Find the electric flux crossing a part of the cylinder defined by ($30° \le \phi \le 60°$) and $0 \le z \le 3.6$ m.

Ans: Flux density on the surface of the cylinder,

$$D_r = \frac{\rho_s}{2\pi r} u_r$$

Differential area on the surface of the cylinder,
$dS = r d\phi dz \, u_r$

Flux crossing the given part of the cylinder,

$$\psi = \int_s D\cdot dS = \int_{\phi=\pi/6}^{\pi/3}\int_0^{3.6} \frac{\rho_s d\phi dz}{2\pi}$$

$$= \left(\frac{8\times10^{-9}}{2\pi}\right)(3.6)\left(\frac{\pi}{6}\right) = 2.4 \, nC$$

Ques 12) Give the statement of Gauss law and define its value for integral and point form.

Or

State point form of gauss's law.

Ans: Gauss' Law (Maxwell's Equation)

Gauss' law, also known as **Gauss' flux theorem**, states that the total electric displacement or electric flux through any closed surface surrounding charges is equal to the net positive charge enclosed by that surface.

Let's consider a point charge Q located in a homogeneous isotropic medium of permittivity, ε. The electric field intensity at any point at a distance r from the charge will be as follows:

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r \qquad \text{..... (1)}$$

And the electric flux density is given as,

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \qquad \text{..... (2)}$$

Now, the electric flux through some elementary surface area dS as shown in **figure 2.8** is

$$d\psi = D dS \cos\theta \qquad \text{.....(3)}$$

Where, θ is the angle between $\vec{D}$ and the normal to dS.

From **figure 2.8,** dS cos θ is the projection of dS normal to the radius vector. By definition of a solid angle,

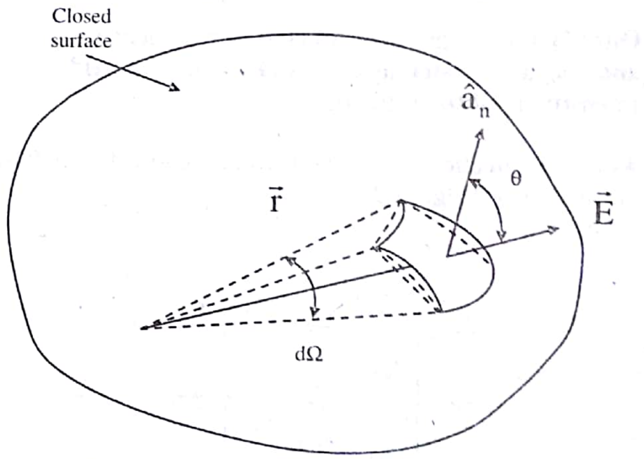

Figure 2.8: Determination of Net Electric Flux through a Closed Surface

$$dS\cos\theta = r^2 d\Omega \qquad \text{.....(4)}$$

Where, dΩ is the solid angle subtended at Q by the elementary surface of area dS.

Thus, total displacement or flux through the entire surface is

$$\psi = \oint_S d\psi = \oint_S D dS\cos\theta = \oint_S D r^2 d\Omega = \frac{Q}{4\pi}\oint d\Omega$$

[Using equations (1), (2), (3) and (4)]

However, from the concept of calculus, the solid angle subtended by any closed surface is 4π steradian. Hence, total displacement or flux passing through the entire surface is,

$$\psi = \oint_S \vec{D}.d\vec{S} = Q = \int_V \rho dv \quad \text{..... (5)}$$

This is the integral form of Gauss' law.

Applying divergence theorem in equation (5), we get

$$\oint_S \vec{D}.d\vec{S} = \int_V \rho dv$$

$$\Rightarrow \int_V (\nabla.\vec{D})dv = \int_V \rho dv$$

$$\Rightarrow \nabla.\vec{D} = \rho$$

$$\nabla.\vec{D} = \rho \quad \text{..... (6)}$$

Equation (6) is the **differential form** or **point form of Gauss' law.**

Ques 13) Define the application of gauss law.

Or

Find electric field density for infinite line charge using gauss's law.

Ans: Application of Gauss Law

The application of gauss law explained below;

1) **Coulomb's Law:** Coulomb's law can be derived from Gauss's law.

Consider electric field of single isolated positive charge of magnitude q as shown below in the **figure 2.9**:

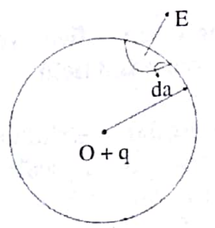

Figure 2.9

Derivation of Coulomb's Law from Gauss's Law

Field of a positive charge is in radially outward direction everywhere and magnitude of electric field intensity is same for all points at a distance r from the charge.

We can assume Gaussian surface to be a sphere of radius r enclosing the charge q.

From Gauss's law

$$\oint E.da = E\oint da = \frac{q_{enc}}{\varepsilon_0}$$

Since E is constant at all points on the surface therefore,

$$EA = \frac{q}{\varepsilon_0}$$

Or,

$$E = \frac{q}{\varepsilon_0 A}$$

Surface area of the sphere is $A = 4\pi r^2$

Thus,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Now force acting on point charge q' at distance r from point charge q is:

$$F = q'E$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

This is nothing but the mathematical statement of Coulomb's law.

2) **Electric Field Due to Infinite Line Charge:** Consider a long thin uniformly charged wire and find the electric field intensity due to the wire at any point at perpendicular distance from the wire.

If the wire is very long and at point far away from both its ends then field lines outside the wire are radial and would lie on a plane perpendicular to the wire.

Electric field intensity have same magnitude at all points which are at same distance from the line charge.

We can assume Gaussian surface to be a right circular cylinder of radius r and length ℓ with its ends perpendicular to the wire as shown below in the **Figure 2.10**:

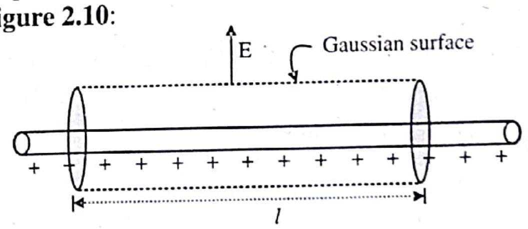

Figure 2.10: Cylindrical Gaussian Surface for Calculation of Electric Field Due to Line Charge

λ is the charge per unit length on the wire. Direction of E is perpendicular to the wire and components of E normal to end faces of cylinder makes no contribution to electric flux. Thus from Gauss's law

$$\oint E.da = \frac{q_{enc}}{\varepsilon_0}$$

Now consider left hand side of Gauss's law

$$\oint E.da = E\oint da$$

Since at all points on the curved surface E is constant. Surface area of cylinder of radius r and length ℓ is $A = 2\pi r\ell$ therefore,

$$\oint E.da = E(2\pi r\ell)$$

Charge enclosed in cylinder q = Linear charge density x Length of cylinder, or $q = \rho\ell$

From Gauss's law

$$\oint E.da = \frac{q}{\varepsilon_0}$$

$$\text{or, } E(2\pi r\ell) = \frac{\rho\ell}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\rho}{2\pi r\varepsilon_0}$$

$$\Rightarrow \quad E \propto \frac{\rho}{r}$$

Thus electric field intensity of a long positively charged wire does not depends on length of the wire but on the radial distance r of points from the wire.

Ques 14) Describe electric potential for a point charge.

Ans: Electric Potential

Electric Potential at a point due to a fixed charge is defined as the work done in bringing one coulomb of charge from infinity to the point against the force created by the fixed charge, i.e., the potential is the work done per unit charge.

The potential, V at a point due to a fixed charge, Q_f is given by,

$$V = \frac{\text{Work done to bring a charge, Q from } \infty \text{ to the point towards } Q_f}{Q}$$

$$\text{Simply} \quad V = \frac{\text{Work Done}}{Q}, \text{ J/C or volt}$$

The potential at a point due to a point charge is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

Ques 15) What do you mean by the potential gradient?

Ans: Potential Gradient

Consider an electric field $\vec{E}$ due to a positive charge placed at the origin of a sphere. Then,

$$V = -\int \vec{E} \cdot d\vec{L} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential decreases as distance of point from the charge increases. This is shown in the **figure 2.11**.

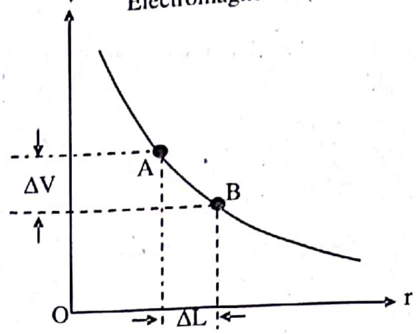

Figure 2.11: Potential Gradient

It is known that the line integral of $\vec{E}$ between the two points gives a potential difference between the two points. For an elementary length ΔL we can write,

$$\therefore V_{AB} = \Delta V = -\vec{E} \cdot \Delta\vec{L}$$

Hence an inverse relation namely the change of potential ΔV, along the elementary length ΔL must be related to $\vec{E}$, as $\Delta L \to 0$.

The rate of change of potential with respect to the distance is called the **potential gradient.**

$$\frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \textbf{Potential gradient}$$

Potential gradient is nothing but the slope of the graph of potential against distance at a point where elementary length is considered.

Ques 16) Prove the electric field vector E = (grad v), where v is a scalar potential field.

Ans: Electric Field Equal to Gradient of Potential

The electric potential is a scalar function for the description of the electrostatic field. It is equal to the work done by the electric field in moving a small charge from an arbitrary point A in the field to a "reference" point, .R, per unit charge:

$$V_A = \int_A^R E \cdot d\ell \text{ (Volts – V)} \qquad .....(1)$$

From equation (1) combined with the law of conservation of energy,

$$\oint_c E \cdot d\ell = 0 \qquad .....(2)$$

The basic expression for the potential

$$V_{(r)} = \frac{Q}{4\pi\epsilon_0 r} \text{ (Reference point at infinity) V} \qquad .....(3)$$

The potential of a given distribution of volume, surface or line charges at a point P of the field,

$$V_P = \frac{1}{4\pi\epsilon_0} \int_v \frac{edv}{r}$$

$$V_p = \frac{1}{4\pi\epsilon_0}\int_s \frac{\sigma ds}{r}$$

$$V_p = \frac{1}{4\pi\epsilon_0}\int_L \frac{Q' d\ell}{r} V \quad .....(4)$$

The potential difference or voltage in the electrostatic field is defined as field is defined as:

$$V_{AB} = V_A - V_B = \int_A^B E \cdot d\ell\, V \quad .....(5)$$

⇒ Hence from equation (5)

The electric field strength is obtained from the electric scalar potential as:

$E = -\text{grad } V = \quad -\nabla.V$

In rectangular form,

$$E = -\text{grad } V = -\nabla \cdot V = -\left(\frac{\partial v}{\partial x}V_x + \frac{\partial v}{\partial y}V_y + \frac{\partial v}{\partial z}V_z\right) V/m$$

Ques 17) Discuss the electric dipole for the point charges.

Ans: Electric Dipole

Two equal and opposite point charge separated by a distance, which is small compared to the distance at which the potential or the field is to be calculated, constitute an electric dipole.

Figure 2.12 show an electric dipole with +q and –q charge separated by a distance d.

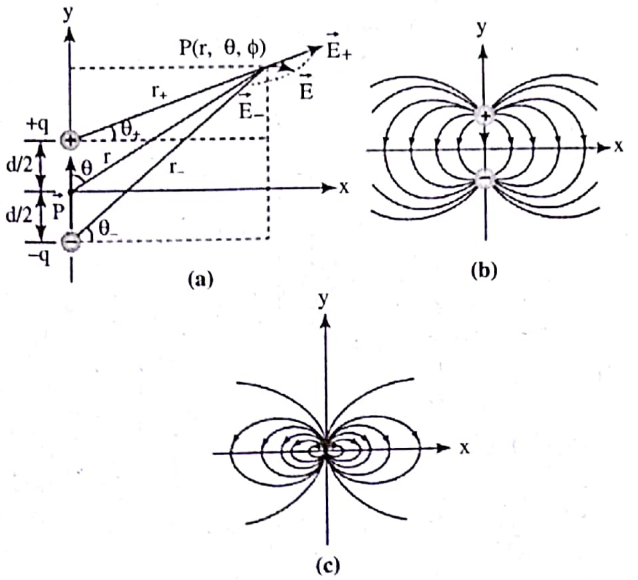

Figure 2.12 (a) Electric Dipole, (b) Field Lines for an Electric Dipole, and (c) Field Lines for a Pure Electric Dipole

Let us compute the electric field intensity at point P. By evaluating the negative gradient of a scalar potential V in spherical coordinates,

$$\vec{E} = \frac{P}{4\pi\varepsilon_0 r^3}[2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta] \quad .....(1)$$

However,

$$2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta = 3\cos\theta\vec{a}_r - (\cos\theta\vec{a}_r - \sin\theta\vec{a}_\theta)$$
$$= 3\cos\theta\vec{a}_r - \vec{a}_z \quad .....(2)$$

Thus, the electric field intensity at point P is given as,

$$\vec{E} = \frac{3(\vec{p}.\vec{r})\vec{r} - r^2\vec{p}}{4\pi\epsilon_0 r^5} \quad .....(3)$$

The electric field intensity falls off as the inverse cube of the distance. In the bisecting plane, $\theta = \pm\pi/2$, the field lines are directed along $\vec{a}_\theta = -\vec{a}_z$. That is,

$$\vec{E} = \frac{\vec{p}}{4\pi\epsilon_0 r^3} \text{ for } \theta = \pm\pi/2 \quad .....(4)$$

However, when $\theta = 0$ or π, the field lines are parallel to the dipole moment $\vec{p}$.

The **concept of an electric dipole is very useful** in explaining the behaviour of an insulting (dielectric) material when it is placed in an electric field. Therefore, a formal definition is in order.

An electric dipole is defined as two charge of equal strength but of opposite polarity but separated by a small distance. Associated with each dipole is a vector called the **dipole moment**.

If q is the magnitude of each charge and $\vec{d}$ is the distance vector from the negative to the positive charge, then the dipole moment is $\vec{p} = q\vec{d}$.

Ques 18) Discuss the equipotential surface? Also draw the equipotential surfaces due to an electric dipole. Locate the points where the potential due to the dipole is zero.

Ans: Equipotential Surfaces

An equipotential surface is a surface with a constant value of potential at all points on the surface. For a single charge q, the potential is given by,

$$V = \frac{1}{4\Delta\varepsilon_0}\frac{q}{r}$$

This shows that V is a constant if r is constant. Thus, equipotential surfaces of a single point charge are concentric spherical surfaces centre at the charge.

No work is done in moving from one point to another in equipotential surface.

An equipotential surface (or simply an equipotential) is a surface that joints points of equal potential.

Since along an equipotential surface $\Delta V = 0$, we conclude $W_{ext} = q\Delta V = 0$. This means that no work is done in moving a charge from one point to another along an equipotential surface.

For a point charge q, electric potential $V = (1/4\pi\varepsilon_0)(q/r)$. Thus, for a given r, V is the same, and the equipotential surfaces are spherical surfaces centred on the charge (**figure 2.13[a]**).

Electric field lines created by any system of charges are always perpendicular to the associated equipotentials (**figure 2.13**). This must be so, because if the field at any point had a component parallel to the equipotential surface through that point, it would require work $W_{ext}(\neq 0)$ to move a test charge q along a constant potential surface ($\Delta V = 0$), which is not true.

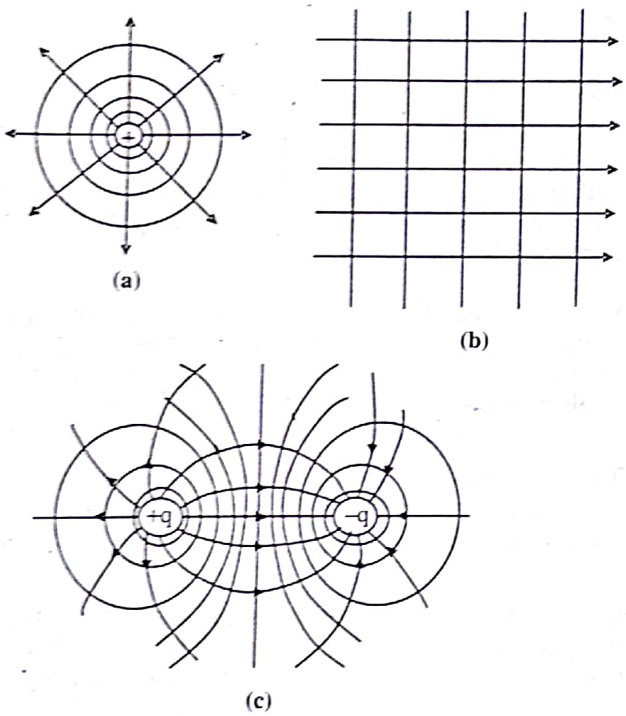

Figure 2.13: Equipotential Surfaces and Field Lines for (a) Point Charge, (b) Uniform Electric Field, and (c) Electric Dipole

Electric potential is zero at all points in the plane passing through the dipole equator.

Ques 19) What do you mean by capacitance?

Ans: Capacitance

Capacitors consists of two parallel conductive plates (usually a metal) which are prevented from touching each other (separated) by an insulating material called the "dielectric".

When a voltage is applied to these plates an electrical current flows charging up one plate with a positive charge with respect to the supply voltage and the other plate with an equal and opposite negative charge.

Then, a capacitor has the ability of being able to store an electrical charge Q (units in Coulombs) of electrons. When a capacitor is fully charged there is a potential difference, p.d., between its plates, and the larger the area of the plates and/or the smaller the distance between them (known as separation) the greater will be the charge that the capacitor can hold and the greater will be its capacitance.

The capacitors ability to store this electrical charge (Q) between its plates is proportional to the applied voltage, V for a capacitor of known capacitance in Farads. Note the capacitance C is always positive and never negative.

The greater the applied voltage the greater will be the charge stored on the plates of the capacitor. The smaller the applied voltage the smaller the charge, therefore, the actual charge Q on the plates of the capacitor and can be calculated as:

Charge on a capacitor, $Q = C \times V$

Where, Q (Charge in Coulombs) = C (Capacitance in Farads) × V (Voltage, in Volts)

Ques 20) Derive the expressions of capacitance for a coaxial cable.

Ans: Capacitance of Coaxial Cable

Let ρ_1 and ρ_2 be the radii of the inner and outer conductors of the coaxial cable. Let ρ_L be the line charge density on the inner conductor and $-\rho_L$ be that on the outer conductor. Then the electric field in the radial direction is

$$E = \frac{\rho_L}{2\pi \in \rho} a_\rho$$

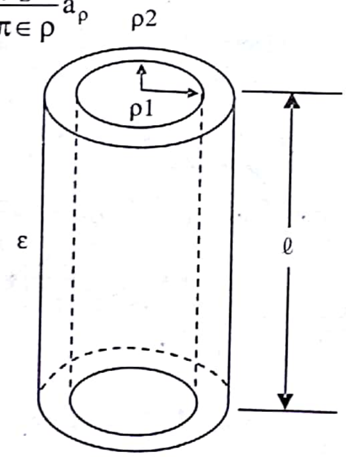

Figure 2.14: Coaxial Cable

The potential difference between the cylinders is

$$V = -\int_{\rho_2}^{\rho_1} E.d\rho a_\rho = -\int_{\rho_2}^{\rho_1} \frac{\rho_L}{2\pi \in \rho} a_\rho \cdot d\rho a_\rho$$

$$\text{Or,} \quad V = -\frac{\rho_L}{2\pi \in} \ln\left(\frac{\rho_2}{\rho_1}\right)$$

$$C = \frac{\rho_L}{V} = \frac{2\pi \in}{\ln\left(\frac{\rho_2}{\rho_1}\right)} \text{ (Farads/m)}$$

The capacitance of a cable of length ℓ metres is

$$C=\frac{2\pi\in\ell}{\ln\left(\frac{\rho_2}{\rho_1}\right)},\ \text{(Farad)}$$

Ques 21) Derive the expressions of capacitance for two wire transmission line.

Ans: Capacitance of Two Wire Transmission Line

Let ρ_L and $-\rho_L$ be the line charge densities of the lines A and B respectively. Let d be the distance between the wires and r be the radius of each wire.

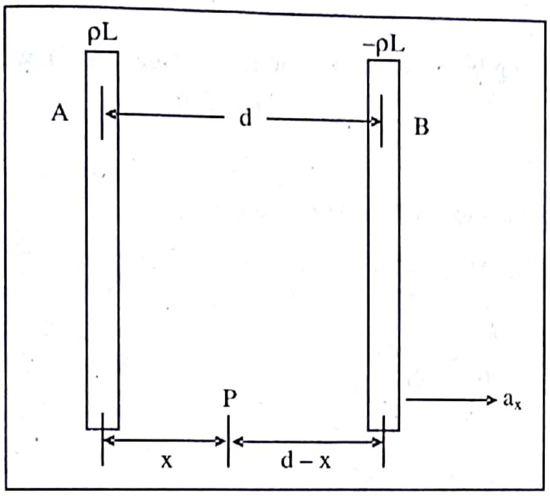

Figure 2.15: Two Wire Transmission Lines

Electric field between the wires at the point P due to ρ_L is

$$E_2=\frac{\rho_L}{2\pi\in x}a_x$$

Electric field at P due to $-\rho_L$ is

$$E_2=+\frac{\rho_L}{2\pi\in(d-x)}a_x$$

The potential difference, V

$$V=-\int_B^A E.dx\,a_x=\int_{(d-r)}^{r}E.dx\,a_x$$

$$=-\int_{(d-r)}^{r}\frac{\rho_L}{2\pi\in x}a_x.dx\,a_x-\int_{(d-r)}^{r}\frac{\rho_L}{2\pi\in(d-x)}a_x.dx\,a_x$$

$$=-\int_{(d-r)}^{r}\frac{\rho_L}{2\pi\in x}dx-\int_{(d-r)}^{r}\frac{\rho_L}{2\pi\in(d-x)}dx$$

$$=-\frac{\rho_L}{2\pi\in}\left[\int_{(d-r)}^{r}\frac{1}{x}dx+\int_{(d-r)}^{r}\frac{1}{(d-x)}dx\right]$$

$$=-\frac{\rho_L}{2\pi\in}\left[\ln\left(\frac{x}{d-r}\right)+\ln\left(\frac{r}{d-r}\right)\right]$$

$$=\frac{\rho_L}{2\pi\in}2\ln\left(\frac{d-r}{r}\right)$$

That is,

$$V=\frac{\rho_L}{\pi\in}\ln\frac{(d-r)}{r}$$

In all practical cases, d >> r. As a result $\frac{(d-r)}{r}\approx\frac{d}{r}$.

So,

$$V=\frac{\rho_L}{\pi\varepsilon}\ln\left(\frac{d}{r}\right)$$

The capacitance,

$$C=\frac{\rho_L}{V}F/m$$

$$C=\frac{\pi\in}{\ln\left(\frac{d}{r}\right)}\ \text{Farad/m}$$

Or capacitance of a pair of wires of length ℓ metres is, therefore, given by

$$C=\frac{\pi\in\ell}{\ln\left(\frac{d}{r}\right)}\ \text{Farad}$$

Ques 22) Define the Poisson and Laplace equation. Also write the Laplace equation in each coordinate system.

Ans: Poisson's and Laplace's Equations

The electric field is related to the charge density by the divergence relationship

$$\nabla\cdot E=\frac{\rho}{\varepsilon_o}$$

E = electric field
ρ = charge density
ε_o = permittivity

and the electric field is related to the electric potential by a gradient relationship,

$$E=-\nabla V$$

Therefore the potential is related to the charge density by **Poisson's equation,**

$$\nabla\cdot\nabla V=\nabla^2V=\frac{-\rho}{\varepsilon_o}$$

In a charge-free region of space, this becomes **Laplace's equation,**

$$\nabla^2V=0$$

This mathematical operation, the divergence of the gradient of a function, is called the **Laplacian**. The Laplace's equation in **Cartesian**, **cylindrical**, and **spherical coordinates** respectively is expressed as,

$$\nabla^2V=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial y^2}=0 \qquad .....(1)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad ....(2)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad ....(3)$$

Ques 23) Using Laplace's equation, obtain the potential distribution between two spherical conductors separated by a single dielectric. The inner spherical conductor of radius, a is at a potential, V_o and the outer conductor of radius, b is at potential zero. Also, find variation of E.

Ans: The Laplace's equation in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

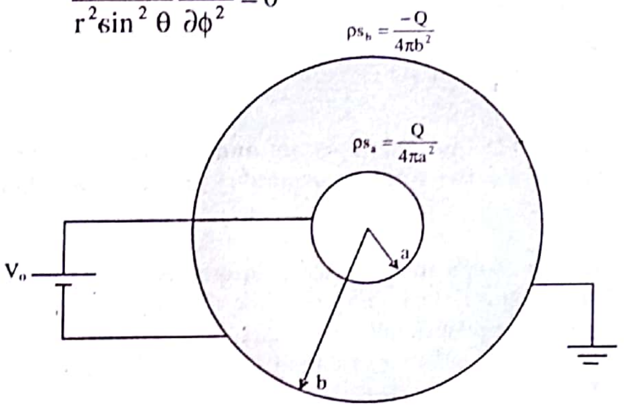

Figure 1.16

Since potential, V is a function of only r, the above equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = 0$$

Since $\frac{1}{r^2} \neq 0$, we get

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = 0$$

Integrating with respect to r, we get

$$r^2\frac{\partial V}{\partial r} = K_1$$

$$\Rightarrow \frac{\partial V}{\partial r} = \frac{K_1}{r^2}$$

Again integrating with respect to r, we get

$$V = -\frac{K_1}{r} + K_2 \quad ....(1)$$

where K_1 and K_2 are the two constants of integration evaluated using suitable boundary conditions.

Boundary conditions:

1) When r = a, $V = V_0$ and
2) When r = b, V = 0

Making use of boundary condition in equation (1), we get

$$V_o = -\frac{K_1}{a} + K_2 \quad ......(2)$$

Substituting boundary condition (2) in equation (2), we get

$$0 = -\frac{K_1}{b} + K_2 \quad ......(3)$$

Solving equations (2) and (3), we get

$$K_1 = -V_o\frac{ab}{(b-a)}$$

$$K_2 = \frac{K_1}{b} = \frac{-V_o a}{(b-a)}$$

Hence,

$$V = \frac{V_o ab}{(b-a)r} - \frac{V_o a}{(b-a)}$$

The relationship between E and V is

$$E = -\nabla V$$

$$\Rightarrow E = -\left[\frac{\partial V}{\partial r}a_r + \frac{1}{r}\frac{\partial V}{\partial \theta}a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}a\phi\right]$$

Since V is a function of r only, the above equation reduces to

$$E = -\frac{\partial V}{\partial r}a_r = -\frac{\partial}{\partial r}\left[\frac{V_o ab}{(b-a)r} - \frac{V_o a}{(b-a)}\right]a_r$$

$$= \frac{V_o ab}{(b-a)r^2}a_r \text{ (V/m)}$$

Hence, $E_r = \frac{V_o ab}{(b-a)r^2}$ (V/m).

Ques 24) Given the electric potential field,
$V = [A\rho^4 + B\rho^{-4}] \sin 4\phi$ (V).
1) **Show that $\nabla^2 V = 0$ in cylindrical coordinates.**
2) **Select the values of a and b so that V = 100 (V) and |E| = 500 (V/m) at P(1,22.5°,2).**

Ans:

1) $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial V}{\partial \rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho(4A\rho^3 - 4B\rho^{-5})\sin 4\phi\right] - \frac{1}{\rho^2}\left[A\rho^4 + B\rho^{-4}\right] \times 16\sin 4\phi + 0$$

$$= \frac{1}{\rho}\left[16\rho^3 A + 16B\rho^{-5}B\right]\sin 4\phi - \frac{1}{\rho^2}\left[A\rho^4 + B\rho^{-4}\right] \times 16\sin 4\phi$$

$$= 0$$

2) $E = -\nabla V$

$$= -\frac{\partial V}{\partial \rho}a_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}a_\phi - \frac{\partial V}{\partial z}a_z$$

$$= -\left[4A\rho^3 - 4B\rho^{-5}\right]\sin 4\phi a_\rho - \frac{1}{\rho}\left[A\rho^4 + B\rho^{-4}\right] \times 4\cos 4\phi a_\phi$$

$\Rightarrow$ $E(1, 22.5°, 2) = -(4A - 4B)a_\rho - 0 \times a_\phi$

$\Rightarrow$ $|E| = \pm[4A - 4B] = 500$

$\Rightarrow$ $4A - 4B = \pm 500$

Also, $V = 100$ (V) at $P(1, 22.5°, 2)$

$\Rightarrow$ $A + B = 100$

Solving, we get,

$A = 112.5$ and $B = -12.5$

Or

$A = -12.5$ and $B = 112.5$

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Module 3

Static Magnetic Field

MAGNETO STATICS

Ques 1) What is a magneto static field? Gives its application.

Ans: Magneto Static Field

Static electric fields are characterised by E or D. Static magnetic fields are characterised by H or B. There are similarities and dissimilarities between electric and magnetic fields. As E and D are related according to $D = \varepsilon E$ for linear material space and H and B are related according to $B = \mu H$.

An electrostatic field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic (or magneto static) field is produced. A magneto static field is produced by a constant current flow (or direct current). This current flow may be due to magnetisation currents as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires.

The development of the motors, transformers, microphones, compasses, telephone bell ringers, television focusing controls, advertising displays, magnetically levitated high speed vehicles, memory stores, magnetic separators, and so on, involve magnetic phenomena and play an important role in our everyday life.

There are two major laws governing magneto static fields:

1) Biot-Savart's law, and
2) Ampere's circuit law.

Application of Magneto Static Field

The applications of magneto static field are as follows:

1) Magnetic separators
2) Particle accelerators like cyclotrons
3) Development of motors
4) Compasses
5) Microphones
6) Telephone ringers
7) Advertising displays
8) Velocity selector
9) Mass spectrometer
10) Transformers
11) Television focus controls
12) High speed velocity devices
13) Magnetohydrodynamic generator

Ques 2) State and explain Biot-savart's law.
Or
State and explain Biot-savart's law.

Ans: Biot-Savart's Law

It states that, "The magnetic flux density of which dB, is directly proportional to the length of the element dl, the current I, the sine of the angle and θ between direction of the current and the vector joining a given point of the field and the current element and is inversely proportional to the square of the distance of the given point from the current element, r."

$$\text{Hence, } dB \propto \frac{Id\ell \sin\theta}{r^2} \text{ or } dB = k\frac{Id\ell \sin\theta}{r^2}$$

Where, k is a constant, depends upon the magnetic properties of the medium and system of the units employed. In SI system of unit,

$$k = \frac{\mu_0\mu_r}{4\pi}$$

Therefore, final Biot-Savart law is;

$$dB = \frac{\mu_0\mu_r}{4\pi} \times \frac{Id\ell \sin\theta}{r^2}$$

Ques 3) Derive magnetic field intensity due to infinitely long wire carrying current I.

Ans: Magnetic Field Intensity Due to Infinitely Long Wire Carrying Current I

Let us consider a long wire carrying a current I and also consider a point P in the space. The wire is presented in the figure 3.1 below, by solid bold lines.

Let us also consider an infinitely small length of the wire $d\ell$ at a distance r from the point P as shown. Here, r is a distance vector which makes an angle θ with the direction of current in the infinitesimal portion of the wire.

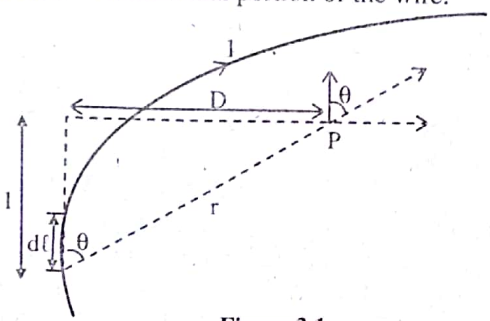


Figure 3.1

If visualises the condition, one can easily understand the magnetic field density at the point P, due to that infinitesimal length dℓ of the wire is directly proportional to current carried by this portion of the wire.

As the current through that infinitesimal length of wire is same as the current carried by the whole wire itself, we can write;

$$dB \propto I \quad .....(1)$$

The magnetic field density at that point P due to that infinitesimal length dℓ of wire is inversely proportional to the square of the straight distance from point P to centre of dℓ. Mathematically, written as;

$$dB \propto \frac{1}{r^2} \quad .....(2)$$

Magnetic field density at that point P due to that infinitesimal portion of wire is also directly proportional to the actual length of the infinitesimal length dℓ of wire.

As θ be the angle between distance vector r and direction of current through this infinitesimal portion of the wire, the component of dℓ directly facing perpendicular to the point P is $d\ell \sin\theta$;

$$dB \propto d\ell \sin\theta \quad .....(3)$$

Now, combining these three statements, we can write;

$$dB \propto \frac{I \cdot d\ell \cdot \sin\theta}{r^2}$$

This is the basic form of **Biot-Savart's Law**.

Thus, in terms of the distributed current sources, the Bio-Savart law as becomes:

$$H = \int_L \frac{Id\ell \times a_R}{4\pi R^2} \quad \text{(Line current)}$$

$$H = \int_S \frac{KdS \times a_R}{4\pi R^2} \quad \text{(Surface current)}$$

$$H = \int_v \frac{Jdv \times a_R}{4\pi R^2} \quad \text{(Volume current)}$$

Ques 4) Derive magnetic field intensity due to finite long wire carrying current I.

Ans: Magnetic Field due to a Straight Current Carrying Conductor of Finite Length

Suppose AB is straight conductor carrying a current of I and magnetic field intensity is to be determined at point P.

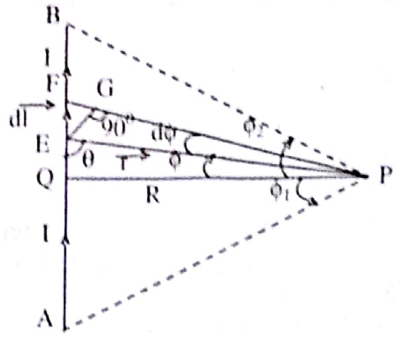


Figure 3.2

According to **Biot-Savart** law the magnetic field at P,

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I\overrightarrow{dl} \times \vec{r}}{r^3}$$

Angle between $I\overrightarrow{dl}$ and $\vec{r}$ is $(180 - \theta)$ so;

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin(180-\theta)}{r^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} \quad ....(1)$$

Now, $EG = EF\sin\theta$
$= dl\sin\theta$

and $EG = EP\sin d\phi = r\sin d\phi$
$= r\,d\phi$

So, $dl\sin\theta = r\,d\phi \quad ....(2)$

So from equation (1),

$$dB = \frac{\mu_0}{4\pi}\frac{Id\phi}{r} \quad ....(3)$$

From Δ EQP, $r = \frac{R}{\cos\phi}$

$$\text{So, } dB = \frac{\mu_0}{4\pi}\frac{I\cos\phi d\phi}{R} \quad ...(4)$$

Then the total magnetic field at point P due to the entire conductor is,

$$B = \int_{-\phi_1}^{\phi_2} \frac{\mu_0}{4\pi}\frac{I}{R}\cos\phi d\phi$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R}[\sin\phi]_{-\phi_1}^{\phi_2}$$

$$B = \frac{\mu_0}{4\pi}\frac{I}{R}(\sin\phi_1 + \sin\phi_2) \quad ...(5)$$

For any conductor of infinite length; $\phi_1 = \phi_2 = 90^0$

So, $B = \frac{\mu_0}{4\pi}\frac{21}{R}$

$$B = \frac{\mu_0 I}{2\pi R} NA^{-1}m^{-1}$$

The direction of magnetic field due to a current carrying conductor can be obtained by using any of the laws like:
1) Right hand palm rule number 1,
2) Right hand thumb rule, or
3) Maxwell right-hand screw rule.

Ques 5) What is amperes force law? Derive the expression for amperes force law.

Ans: Ampere's Force Law
Ampere's Force Law describes the force between conductors that are carrying current. For this Law, let consider the two conductors parallel to one another so that the force is maximum. While this is not "general", most of our magnetic systems involve conductors that are arranged for maximum force.

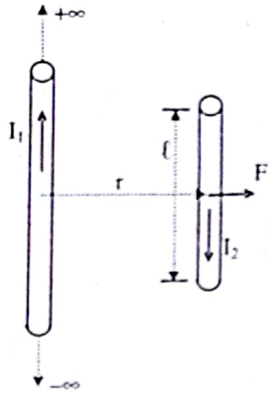


Figure 3.3: Ampere's Force Law

Figure 3.3 shows the basic structure. The conductor on the left is carrying a current, flowing upward, of I_1 amperes. The conductor is infinite in both direction, which is impractical, but as long as the loop is closed "a long way away," "infinite" is OK.

The Ampere's force law gives the magnetic force that exists between two current-carrying current loops. Another version of this law gives the force between the current elements of the different current loops. Both the version of this law are derivable from the Lorentz force law.

When two current loops are oriented in such a way that the field produced by one loop links with the other then each loop will be acted upon by an electromagnetic force. Let us derive expressions for the electromagnetic force between two filamentary loops separated by a finite distance in a homogeneous medium of permeability (μ). The loops denoted by C_1 and C_2 in **figure 3.4** carry currents I_1 and I_2 respectively in the same sense.

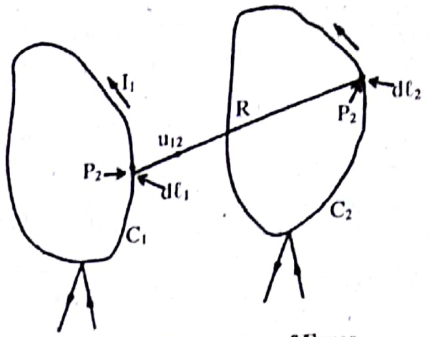


Figure 3.4: For Calculation of Force between Two Current–Carrying Loops

In order to calculate force experienced by one loop due to the magnetic field of the other loop, we at first find the force on a current element of a loop and then integrate it around the loop. Let us take a current element $I_1 d\ell_1$ at point P_1 on loop C_1 and a current-element $I_2 d\ell_2$ at point P_2 on loop C_2. The distance between P_1 and P_2 is taken as R. Let the flux density at P_2 due to $I_1 d\ell_1$ be dB_1.

Then according to Biot-Savart law,

$$dB_1 = \frac{\mu I_1}{4\pi R^2}(d\ell_1 \times u_{12})$$

Where, u_{12} is a unit vector directed from P_1 to P_2. Hence, flux density at P_2 due to the total current in C_1 is given by

$$B_1 = \oint_{C_1} \frac{\mu I_1}{4\pi R^2}(d\ell_1 \times u_{12})$$

The flux density will cause a force dF_2 on the current element $I_2 d\ell_2$. The force as per formula is:

$$dF_2 = I_2(d\ell_2 \times B_1)$$

$$= \frac{\mu I_1 I_2}{4\pi}(d\ell_2 \times \oint_{C_1} \frac{1}{4\pi R^2}(d\ell_1 \times u_{12}) \quad .....(1)$$

Total force on loop C_2 is obtained by integrating equation (1) around the loop. Thus force on the loop,

$$F_2 = \frac{\mu I_1 I_2}{4\pi} \oint_{C_1} d\ell_2 [\oint_{C_1} \frac{1}{4\pi R^2}(d\ell_1 \times u_{12})$$

$$= \frac{\mu I_1 I_2}{4\pi} \oint_{C_2}\oint_{C_1} \frac{1}{4\pi R^2}[d\ell_2 \times (d\ell_1 \times u_{12})] \quad .....(2)$$

The force F_1 on loop C_1 is obtained by integrating the subscripts 1 and 2 in the expression of equation (2). Thus, F_1

$$= \frac{\mu I_1 I_2}{4\pi} \oint_{C_1}\oint_{C_2} \frac{1}{4\pi R^2}[d\ell_2 \times (d\ell_2 \times u_{12})] \quad .....(3)$$

Equation (2) or (3) is known as **Ampere's force law.** The electromagnetic force is proportional to the product of current magnitudes and permeability of the medium. It also depends on size and shape of the loops and distance between them.

Ques 6) A rectangular loop of length ℓ and width w carries a steady current I_1. The loop is then placed near an infinitely long wire carrying a current I_2, as shown is figure 3.5. What is the magnetic force experienced by the loop due to the magnetic field of the wire?

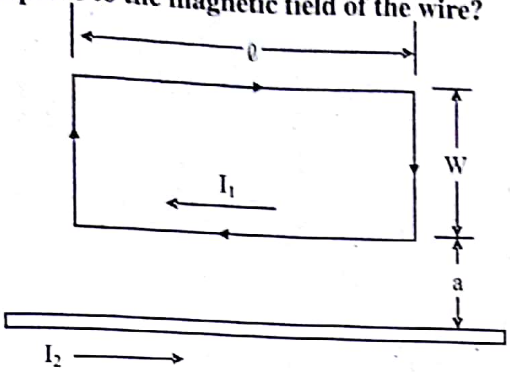


Figure 3.5: Magnetic Forces on a Current Loop

Ans: The forces are shown in **figure 3.6.** The magnetic induction due to the infinitely long wire is,

$$\bar{B}_2 = \frac{\mu I_2}{2\pi r}\hat{a}_\phi$$

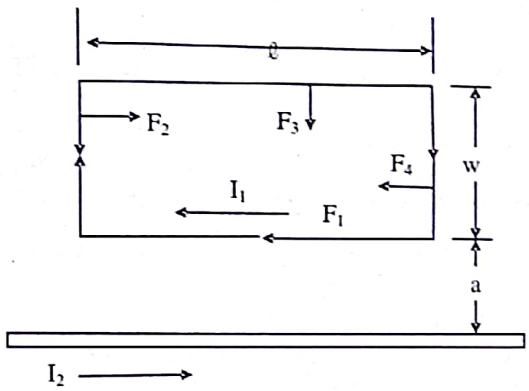


Figure 3.6: Magnetic Forces on the Loop and the Wire

The force on the loop is given as:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$

Here, $\vec{F}_1 = I_1\int d\ell_1 \times \vec{B}_2 = I_1\int_{z=l}^{0} dz\hat{a}_z \times \frac{\mu I_2}{2\pi r}\hat{a}_\phi\Big|_{r=a} = \frac{\mu I_1 I_2 \ell}{2\pi a}\hat{a}_r$

(Repulsive)

$$\vec{F}_3 = I_1\int \overrightarrow{d\ell_3} \times \vec{B}_2 = I_1\int_{z=0}^{l} dz\hat{a}_z \times \frac{\mu I_2}{2\pi r}\hat{a}_\phi\Big|_{r=a+w} = \frac{\mu I_1 I_2 \ell}{2\pi a(a+w)}\hat{a}_r$$

(Attractive)

$$\vec{F}_2 = I_1\int \overrightarrow{d\ell_2} \times \vec{B}_2 = I_1\int_{r=0}^{w} dr\hat{a}_r \times \frac{\mu I_2}{2\pi r}\hat{a}_\phi = \frac{\mu I_1 I_2 \ell}{2\pi}\ln w\hat{a}_z$$

(Parallel)

$$\vec{F}_4 = I_1\int \overrightarrow{d\ell_2} \times \vec{B}_2 = I_1\int_{r=w}^{0} dr\hat{a}_r \times \frac{\mu I_2}{2\pi r}\hat{a}_\phi = -\frac{\mu I_1 I_2}{2\pi}\ln w\hat{a}_z$$

(Parallel)

Thus, the total force on the loop is

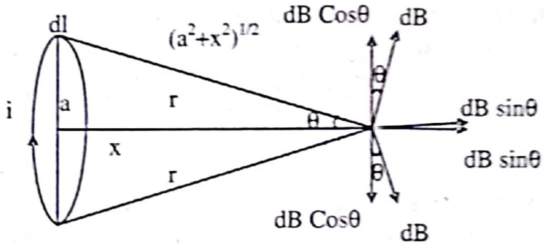
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \frac{\mu I_1 I_2 \ell}{2\pi a}\hat{a}_r - \frac{\mu I_1 I_2 \ell}{2\pi(a+w)}\hat{a}_r$$

$$= \frac{\mu I_1 I_2 \ell}{2\pi}\left[\frac{1}{a} - \frac{1}{a+w}\right]\hat{a}_r$$

Ques 7) Derive the magnetic field intensity on the axis of a circular loop carrying current I.

Ans: Magnetic Field Intensity on the Axis of a Circular Loop

Consider a circular conducting coil of radius 'a' carrying current i. The loop lies on yz plane and its axis lies on x axis. Let us derive field at point P at a distance x from the centre. Consider a small element at dl on the coil.



$$dB = \frac{\mu_0}{4\pi}\frac{idl\sin\varphi}{(a^2+a^2)}$$

Figure 3.7

As the loop lies perpendicular to the plane of paper and vector $\vec{r}$ in the plane of paper, hence angle ϕ between $\overrightarrow{dl}$ and $\vec{r}$ is 90^0

$$\therefore dB = \frac{\mu_0 idl}{4\pi(a^2+x^2)}$$

Magnetic field $\overrightarrow{dB}$ can be resolved into two components one dBsinθ parallel to the axis of the loop and other dBcosθ perpendicular to the axis.

From the symmetry of the system it can be seen that diametrically opposite elements contribute to cancel the perpendicular components whereas parallel components are added up.

$$B = \int dB\sin\theta$$

Thus,

$$B = \oint \frac{\mu_0}{4\pi}\frac{idl}{r^2}\sin\theta$$

From the diagram we can observe:

$$r = \sqrt{(a^2+x^2} \text{ and } \sin\theta = a/\sqrt{(a^2+x^2}$$

$$\therefore dB = \oint \frac{\mu_0}{4\pi}\frac{idl}{(a^2+x^2)}\frac{a}{(a^2+x^2)^{1/2}}$$

$$B = \frac{\mu_0 ia}{4\pi(a^2+x^2)^{3/2}} \oint dl$$

$$B = \frac{\mu_0 ia}{4\pi(a^2+x^2)^{3/2}} 2\pi a$$

As we know area of circular coil is:
$A = \pi a^2$

$$\therefore B = \frac{\mu_0}{4\pi} \frac{2iA}{(a^2+x^2)^{3/2}}$$

For coil with N turns:

$$\therefore B = \frac{\mu_0}{4\pi} \frac{2NiA}{(a^2+x^2)^{3/2}}$$

We have magnetic dipole moment of coil;
$M = Nia$

$$\therefore B = \frac{\mu_0}{4\pi} \frac{2M}{(a^2+x^2)^{3/2}}$$

Ques 8) The mean radius of a circular coil of 50 turns of fine wire is 8.0 cm, It carries a current of 3.0 A. The coil is located on the y–z plane in air. Find the magnetic field intensity vector at P(20 cm.0,0).

Ans: Number of turn, N = 50
Radius of the coil, a = 0.08 m
Current, I = 3.0 A

Distance of P from the centre of the coil, h = 0.2m

Flux density,

$$B = \frac{\mu_0 (NI) a^2}{2(a^2+h^2)^{3/2}} u_x = \frac{(4\pi\times10^{-7})(50\times3)(0.08)^2}{2(0.08^2+0.2^2)^{3/2}} u_x$$

$$= 603.49\times10^{-7} u_x T$$

Ques 9) Derive the magnetic field intensity on the axis of a rectangular loop carrying current I.

Ans: Rectangular Current Loop
A rectangular loop with side length 2a and 2b, and carrying I, as illustrated in **figure 3.8**. The flux density on the axis of the loop using the vector from of Biot-Savart law using rectangular coordinates can be calculated.

The loop is taken on the x–y plane with its sides parallel to x and y axes. The centre of the loop is at the origin. Point P at which we wish to find the field is taken at a height h from the plane of the loop. Thus, the coordinates of P are (0, 0, h).

Now let us take a differential length on side 1 at P_1. The coordinates of P_1 are (x, b, 0). A differential current element vector at P_1 is

$$I d\ell = -I dx\, u_x,$$

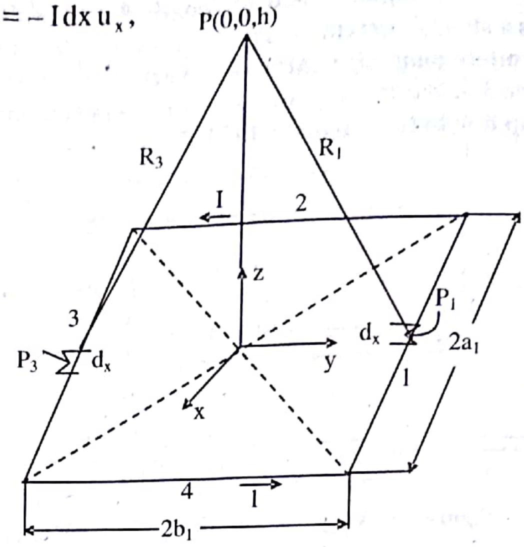


Figure 3.8: Rectangular Current Loop on the x–y Plane

The minus sign is used because I and dx are in opposite directions.

Referring to the **figure 3.8**, displacement of P from P_1 is:

$$R_1 = -xu_x - bu_y + hu_z$$

Flux density at P due to the current element,

$$dB_1 = -\frac{\mu I}{4\pi R_1^3}(d\ell \times R_1) = -\frac{\mu I}{4\pi R_1^3}[dx u_x \times (-xu_x - bu_y + hu_z)]$$

$$= \frac{\mu I dx}{4\pi R_1^3}(hu_y + bu_z)$$

Where, $R_1^3 = (x^2+b^2+h^2)^{3/2}$

It can similarly be shown that the flux density at P due to the current element at P_3(x, –b, 0) on side 3 is given by,

$$= \frac{\mu I dx}{4\pi R_3^3}(-hu_y + bu_z)$$

Where, $R_3 = R_1$

The sum of dB_1 and dB_3 which we well denote by $dB_{(1+3)}$, is,

$$dB_{(1+3)} = \frac{\mu b I dx}{2\pi R_1^3} u_z$$

Thus, the total field due to currents in sides 1 and 3,

$$B_{(1+3)} = u_z \int_{-a}^{a} \frac{\mu b I}{2\pi R_1^3} dx$$

$$= \frac{\mu a b I}{\pi(b^2+h^2)\sqrt{a^2+b^2+h^2}} u_z \quad ....(1)$$

Similarly, it can be shown that flux density at P due to currents in sides 2 and 4,

$$B_{(2+4)} = \frac{\mu abI}{\pi(b^2+h^2)\sqrt{a^2+b^2+h^2}} u_z \quad ....(2)$$

The flux density B on the axis and at a height h from the plane of the loop is equal to the sum of equation (1) and (2),

Thus,

$$B = \frac{\mu abI}{\pi\sqrt{a^2+b^2+h^2}}\left(\frac{1}{a^2+h^2}+\frac{1}{b^2+h^2}\right)u_z \quad ....(3)$$

If the loop has N turns, the flux density is given by,

$$B = \frac{\mu NabI}{\pi\sqrt{a^2+b^2+h^2}}\left(\frac{1}{a^2+h^2}+\frac{1}{b^2+h^2}\right)u_z \quad ....(4)$$

By substituting h = 0 in equations (3) and (4), we will obtain the flux density at the centre of the loop. It is given by,

$$B(h=0) = \frac{\mu NI\sqrt{a^2+b^2}}{\pi ab} u_z \quad ....(5)$$

The fields are along the axial direction. It may be noted that the side length of the rectangular loop are 2a and 2b.

Ques 10) What is magnetic vector potential? Derive the relation between scalar potential and vector potential.

Ans: Magnetic Vector Potential
Divergence of the curl of any vector field is identically zero, i.e.,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \text{ for any vector field } \vec{A} \quad .....(1)$$

Because the fourth of Maxwell's equations states that $\vec{B}$ is solenoid, as given by $(\vec{\nabla}\cdot\vec{B}=0)$, we can thus assume that $\vec{B}$ may be written in terms of another vector field, $\vec{A}$, that we will call the magnetic vector potential.

$$B = \vec{\nabla}\times\vec{A} \quad .....(2)$$

Note: From equation (1) that, given a magnetic flux density, $\vec{B}$, there we will be an infinite number of vector fields, $\vec{A}$, that can satisfy the identity, e.g., adding a constant to $\vec{A}$ will also satisfy equation (2). This means that, to specify a unique definition of the vector field, $\vec{A}$, we will need to make an additional restriction on $\vec{A}$. The additional restriction is called a gauge and it is arbitrary.

Substituting equation (2) into Faraday's law $(\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t)$, we can write,

$$\vec{\nabla}\times\vec{E} = -\partial(\vec{\nabla}\times\vec{A})/\partial t$$

or,
$$\vec{\nabla}\times(\vec{E}+\partial\vec{A}/\partial t) = 0 \quad .....(3)$$

Since, $\vec{\nabla}\times(-\vec{\nabla}V) = 0$ for any scalar field. Thus, because the curl of the vector field shown in parenthesis in equation (3) is zero (i.e., it is irrotational), then that field can be written as the negative gradient of another scalar field, V, that we will call the electric scalar potential.

$$\vec{E}+\partial\vec{A}/\partial t = -\vec{\nabla}V$$

or,
$$\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t \quad .....(4)$$

Note: From equation (4) that the electric field intensity, $\vec{E}$, can be written in terms of the electric scalar potential, V, and the magnetic vector potential, $\vec{A}$. As long as these two potentials are unique, the electric field intensity will also be unique.

In the special case of static (time-independent) fields and potentials, $\partial\vec{A}/\partial t = 0$, and we can see that equation (4) reduces to $\vec{E} = -\vec{\nabla}V$.

Ques 11) Explain magnetic flux density/Maxwell equation.

Ans: Magnetic Flux Density-Maxwell equation
The magnetic flux density B is similar to the elecrtic flux density D. Therefore, the magnetic flux density B is related to the magnetic field intensity H

$$B = \mu_o H$$

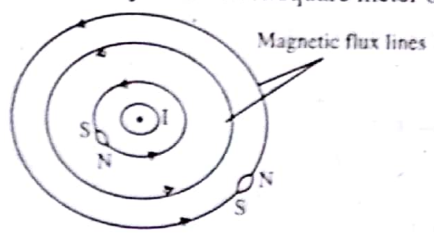
Where, μ_o is a constant and is known as the permeability of free space. Its unit is Henry/meter (H/m) and has the value,

$$\mu_o = 4\pi\times10^{-7} \text{H/m}$$

The magnetic flux through a surface S is given by,

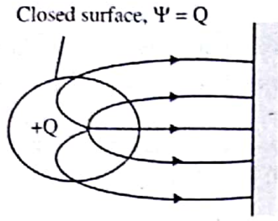
$$\psi = \int_S B\cdot dS$$

Where the magnetic flux ψ is in webers (Wb) and the magnetic flux density is in weber/square meter or Teslas.

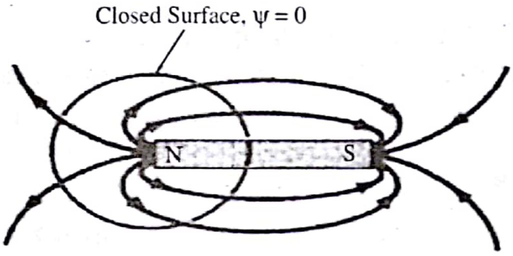


Magnetic flux lines due to a straight wire with current coming out of the page. Each magnetic flux line is closed with no beginning and no end and are also not crossing each other. In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed.

$$\psi = \oint D \cdot dS = Q$$



Thus it is possible to have an isolated electric charge. Also the electric flux lines are not necessarily closed.
Magnetic flux lines are always close upon themselves.



So it is not possible to have an isolated magnetic pole (or magnetic charges).An isolated magnetic charge does not exist. Thus the total flux through a closed surface in a magnetic field must be zero.

$$\oint B \cdot dS = 0$$

This equation is known as the law of conservation of magnetic flux or Gauss's Law for magneto static fields.

Magneto static field is not conservative but magnetic flux is conserved.

Applying Divergence theorem, we get,

$$\oint_S B \cdot dS = \oint_v \nabla \cdot B\, dv = 0$$

Or $\nabla \cdot B = 0$

This is Maxwell's fourth equation.

This equation suggests that magneto static fields have no source or sinks. Also magnetic flux lines are always continuous.

Ques 12) Solve the following:

a) A radial field, $\vec{H} = \frac{2.39\times10^6}{r}\cos\varphi\hat{a}_r$ A/m exists in free space. Find magnetic flux ϕ crossing the surface defined by $-\pi/4 \le \phi \le \pi/4$, $0 \le z \le 1$m.

b) Compute the total magnetic flux ϕ crossing the z = 0 plane in cylindrical coordinates for $r \le 5\times10^{-2}$m if, $\vec{B} = \frac{0.2}{r}\sin^2\varphi\hat{a}_r (T)$.

Ans:

a) $\vec{H} = \frac{2.39\times10^6}{r}\cos\phi\hat{a}_r$

$\because$ Magnetic flux crossing the given surface is given as:

$$\phi = \int \vec{B}\cdot\vec{dS} = \int \mu_0 \vec{H}\cdot\vec{dS}$$

$$= \int_0^1 \int_{-\pi/4}^{\pi/4} \left(\frac{2.39\times10^6}{r}\mu_0 \cos\phi\hat{a}_r\right)(rd\phi dz\hat{a}_r)$$

$$= \int_0^1 \int_{-\pi/4}^{\pi/4} \frac{2.39\times10^6}{r} \times 4\pi\times10^{-7} \times r\cos\phi d\phi dz$$

$$= \int_0^1 \int_{-\pi/4}^{\pi/4} 3\cos\phi d\phi dz$$

$$= 3[\sin\phi]_{-\pi/4}^{\pi/4} = 3\sqrt{2} = 4.24\,\text{Wb}$$

b) $\vec{B} = \frac{0.2}{r}\sin^2\phi\hat{a}_r$

Magnetic flux crossing the given surface is given as:

$$\phi = \int \vec{B}\cdot\vec{dS} = \int_0^{5\times10^{-2}} \int_0^{2\pi} \left(\frac{0.2}{r}\sin^2\phi\hat{a}_z\right)(rdrd\phi\hat{a}_z)$$

$$= 0.2\int_0^{5\times10^{-2}} \int_0^{2\pi} \sin^2\phi dr d\phi = 0.2\times5\times^{10^{-2}} \int_0^{2\pi} \sin^2 d\phi$$

$$= 10^{-2}\int_0^{2\pi} \frac{1-\cos 2\phi}{2} d\phi$$

$$= 10^{-2}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{2\pi}$$

$$= 10^{-2}[\pi - 0] = 3.14\times10^{-2}\,\text{Wb}$$

Ques 13) Define Ampere's current law/Maxwell's equation for time varying field.

Or

State Ampere's circuital Law. Also prove that the integral $\overline{H}\cdot d\overline{L}$ along the closed path gives the direct current enclosed by that closed path.

Ans: Ampere's Circuital/ Ampere's Current Law-Maxwell's equation

Ampere's circuital law in magnetism is analogous to gauss's law in electrostatics. This law is also used to calculate the magnetic field due to any given current distribution. This law states that "The line integral of resultant magnetic field along a closed plane curve is equal to μ_0 time the total current crossing the area bounded by the closed curve provided the electric field inside the loop remains constant".

Thus, $\oint B.d\ell = \mu_0 I_{enc}$ ..... (1)

$$\Rightarrow \oint \frac{\overline{B}}{\mu_0}\cdot\overline{d\ell} = I$$

$$\Rightarrow \oint \overline{H}\cdot\overline{d\ell} = I$$

$$\left[\because \overline{H} = \frac{\overline{B}}{\mu_0}\right]$$

Where, μ_0 is the permeability of free space and I_{enc} is the net current enclosed by the loop as shown below in the **Figure 3.9.**

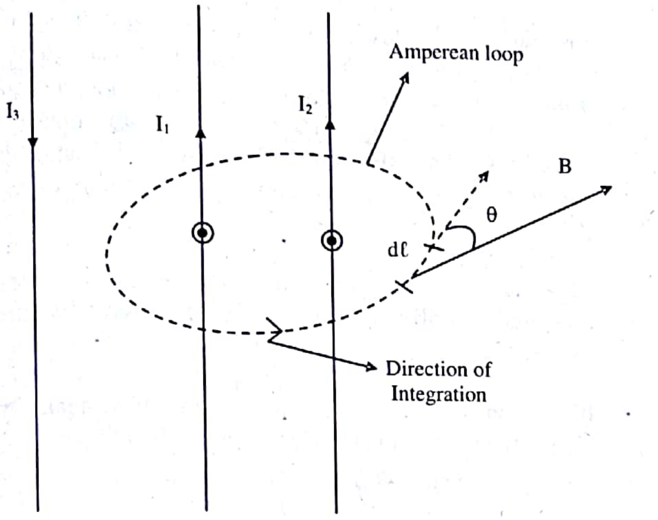


Figure 3.9: Ampere's law Applied to a Loop Containing Two Long Straight Wires

Consider a long straight conductor carrying direct current I placed along z-axis as shown in the **Figure 3.10**. Consider a closed circular path of radius r which encloses the straight conductor carrying direct current I. The point P is at a perpendicular distance r from the conductor. Consider $d\overline{L}$ at point P which is in $\overline{a}_\phi$ direction, tangential to circular path at point P.

$$d\overline{L} = r d\phi \overline{a}_\phi \qquad .....(2)$$

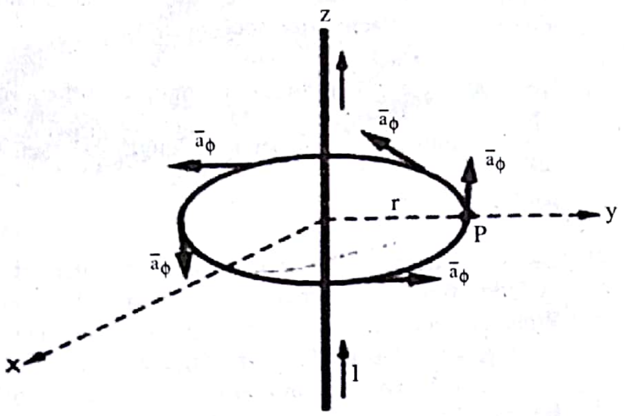


Figure 3.10

While $\overline{H}$ obtained at point P, from Biot-Savart law due to infinitely long conductor is;

$$\overline{H} = \frac{I}{2\pi r}\overline{a}_\phi \qquad .....(3)$$

$$\therefore \quad \overline{H}\cdot d\overline{L} = \frac{I}{2\pi r}\overline{a}_\phi \cdot r d\phi \overline{a}_\phi$$

$$= \frac{I}{2\pi r} r d\phi = \frac{I}{2\pi} d\phi \quad (\overline{a}_\phi \cdot \overline{a}_\phi = 1)$$

Integrating $\overline{H}\cdot d\overline{L}$ over the entire closed path,

$$\oint \overline{H}\cdot d\overline{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi}[\phi]_0^{2\pi} = \frac{I2\pi}{2\pi}$$

$= I =$ Current carried by conductor

This proves that the integral $\overline{H}\cdot d\overline{L}$ along the closed path gives the direct current enclosed by that closed path.

Ques 14) Give the applications of ampere's law.

Ans: Application of Ampere's Law

Ampere's law is frequently used for evaluating $\dot{H}$ or $\dot{B}$ by forming a suitable path known as Amperian path around the current. The Amperian path should either be parallel or perpendicular to field lines at every point.

The current enclosed inside the path only contributes to right side of Ampere's law equation and the current outside the path does not contribute. There are some applications are as follows:

1) **Magnetic Field Produced by a Long Straight Wire Carrying Current I:** Consider a long straight wire carrying a current I as shown in **Figure 3.11.**

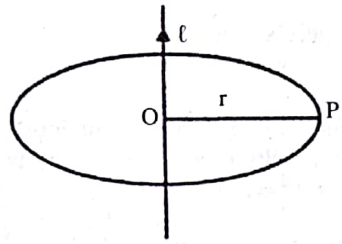


Figure 3.11

To find the expression for magnetic field ar magnetic flux density $\vec{B}$ at a point P, an Amperian surface as a circle of radius r is drawn around the wire passing through P. The magnetic field is tangential to the circle and has the same magnitude at all points around the circle. Hence the integral;

$$\oint \vec{B}\cdot \vec{d\ell} = \oint B d\ell = B\oint d\ell \qquad .....(1)$$

But $\oint d\ell$ around the circle of radius r is $2\pi r$.

Therefore, $\oint \vec{B}\cdot \vec{d\ell} = B \cdot 2\pi r \qquad .....(2)$

From Ampere's law, we have;

$$\oint \vec{B}\cdot \vec{d\ell} = \mu_0 I \qquad .....(3)$$

Equation equations (2) and (3), we get;

$B \cdot 2\pi r = \mu_0 I$

Or $B = \dfrac{\mu_0 I}{2\pi r} \qquad .....(4)$

2) **Magnetic Field Inside a Long Cylindrical Wire: Figure 3.12** shows the cross-section of a cylindrircal wire of radius R carring current I, uniformly distributed throughout its cross-section.

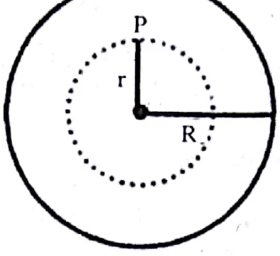


Figure 3.12

Let's use Ampere's law to find the field inside a long straight wire of radius R carrying a current I. Assume the wire has a uniform current per unit area:

$$J = I/\pi R^2$$

To find the magnetic field at a radius r inside the wire, draw a circular loop of radius r. The magnetic field should still go in circular loops, just as it does outside the wire.

Apply Ampere's Law

$$\int B \cdot ds = 2\pi rB = \mu_o I_{enc}$$

The current passing through our loop is the current per unit area multiplied by the area of the loop:

$$I_{enc} = Js\, \pi r^2 = Ir^2/R^2$$

Therefore $2\pi rB = \mu_o\, Ir^2/R^2$

$$B = \mu_o Ir/2\pi R^2$$

So, inside the wire the magnetic field is proportional to r, while outside it is proportional to 1/r.

3) **Magnetic Field Intensity H Due to a Toroid:** Toroid is a hollow circular ring (like a medu vadai) on which a large number of turns of a wire are wound.

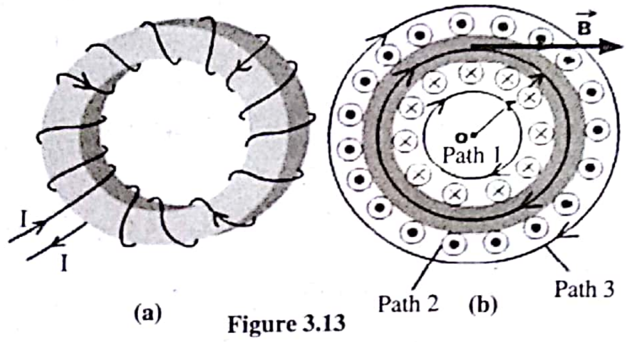


Figure 3.13

The **figure 3.13** represents a toroid wound with a wire carrying a current I. Consider path 1, by symmetry, if there is any field at all in this region, it will be tangent to the path at all point and $\oint B \cdot d\ell$ will equal the product of B and the circumference d = 2pr of the path. The current through the path however is zero and hence from Ampere's law the field B must be zero.

Similarly, if there is any field at path 3, it will also be tangent to the path at all points. Each turn of the winding passes twice through the area bounded by this path, carrying equal currents in opposite directions. The net current through the area is therefore zero and hence B = 0 at all points of the path.

The field of the toroidal solenoid is therefore confined wholly to the space enclosed by the windings.

If we consider path 2, a circle of radius r, again by symmetry the field is tangent to the path, and:

$$\oint B\, d\ell = B.\oint d\ell = B2\pi r$$

Each turn of the winding passes once through the area bounded by path 2 and total current through the area is Ni, where N is the total number of turns in the windings.

Using Ampere's law

$$2\pi rB = \mu_0 Ni$$

$$B = \frac{\mu_0 Ni}{2\pi r}$$

If the radial thickness of the core is small, field is almost constant across the section.

Here $2\pi r$ is the circumferential length of to the toroid.

$\frac{N}{2\pi r}$ – number of rtuns per unit length 'n' then $B = \mu_0 ni$.

4) **Electric Field due to a Solenoid:** A solenoid is a cylinder tightly wrapped with a thin wire over it. When current is passed through the wire, the magnetic field produced outside is negligible and the field inside is uniform and parallel to the axis as shown in **figure 3.14.**

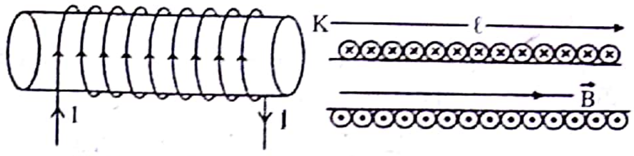


Figure 3.14

The magnetic field is due to a section of the solenoid which has been stretched out for clarity. Only the interior semi-circular part is shown in **figure 3.15**. Notice how the circular loops between neighbouring turns tend to cancel.

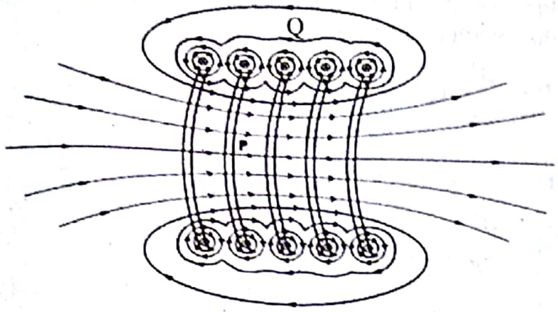


Figure 3.15

Solenoid is long wire wound in form of helix such that the length of solenoid is large compared to the radius of the closely spaced turns.

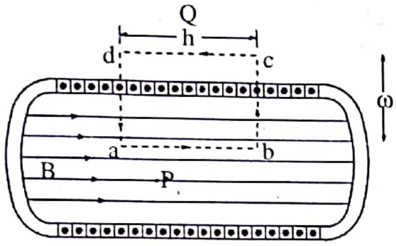


Figure 3.16

In the **figure 3.16** the upper dots represent the current coming out of the paper. Using the right hand rule the fields' lines inside the solenoid go from left to right. Similarly the crosses on the lower side represent the current going into the paper. The field lines inside the solenoid also go from left to right. The two fields being in the same direction add up but outside the solenoid they cancel (i.e., the field contribution due to the dots and crosses).

To find the magnetic field due to a solenoid consider the Amperian loop (imaginary closed path) as shown in the **figure 3.16**.

The field along cd is zero as it is outside the solenoid. Along da and bc the transverse section the field is zero outside the solenoid (also, B is perpendicular to $d\ell$ so $\vec{B}\cdot d\vec{\ell}=0$). Therefore the only contribution is from ab. Let the length ab be 'h'. If there are n turns per unit length, then the enclosed current i_e is

$$i_e = i(nh),$$

Where, Current in the solenoid

$$\therefore \oint \vec{B}\cdot d\vec{\ell} = \mu_o i_e$$

$$\int_{ab} B\, d\ell = \mu_o i_e$$

$$Bh = \mu_o\, nhi \qquad \left(\because \int_{ab} d\ell = ab = h\right)$$

$$B = \mu_o ni$$

Direction is given by the right hand rule.

Ques 15) Find the vector magnetic potential and hence the magnetic flux density $\vec{B}$ due to an infinite wire carrying a current. at a point (i) inside, (ii) outside the wire.

Ans:

1) **Inside the Wire:** Let a be the radius of the wire. By symmetry, it is understood that only the z- component of the vector potential exists.

$$\nabla^2 A_z = -\mu J_z = -\frac{\mu I}{\pi a^2}$$

$$\text{Or, } \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -\frac{\mu I}{\pi a^2}$$

Integrating,

$$r\frac{\partial A_z}{\partial r} = -\frac{\mu I r^2}{2\pi a^2} + C_1$$

$$\text{Since, } r\frac{\partial A_z}{\partial r} = 0 \text{ at } r = 0 \Rightarrow C_1 = 0$$

Integrating again,

$$A_z = -\frac{\mu I r^2}{4\pi a^2} + C_2$$

$$\text{Since } A_z = 0 \text{ at } r = a, \Rightarrow C_2 = \frac{\mu I}{4\pi}$$

$$A_z = \frac{\mu I}{4\pi}\left[1 - \frac{r^2}{a^2}\right]$$

In vector form, the vector magnetic potential is given as:

$$\vec{A} = \frac{\mu I}{4\pi}\left[1 - \frac{r^2}{a^2}\right]\vec{a}_z$$

$$\text{Now, } \vec{B} = \nabla \times \vec{A}$$

$$\therefore B_r = \left(\text{curl}\vec{A}\right)_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} = 0$$

$$B_\phi = \left(\text{curl}\vec{A}\right)_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu I r}{2\pi a^2}$$

$$B_z = \left(\text{curl}\vec{A}\right)_z = \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} = 0$$

Thus, the magnetic induction is:

$$\vec{B} = \frac{\mu I r}{2\pi a^2}\hat{a}_\phi$$

2) Outside the Wire

Here, $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)=0$

$$\Rightarrow \quad \frac{\partial A_z}{\partial r}=\frac{C_1}{r} \quad .....(1)$$

$$A_z = C_1 \ln r + C_2$$

At, $r = a$, $A_z = 0 \Rightarrow C_2 = -C_1 \ln a$

$$\therefore A_z = C_1 \ln\left(\frac{r}{a}\right)$$

The constant C_1 is found from the boundary condition for $\frac{\partial A_z}{\partial r}$ at $r = a$

Since $\vec{B} = \nabla \times \vec{A}$,

$$\therefore B_\phi = -\frac{\partial A_z}{\partial r}$$

Now, B_ϕ must be continuous at $r = a$. From the result of equation (1), we get

$$-\frac{\partial A_z}{\partial r}=\frac{\mu I}{2\pi a} \quad .....(2)$$

From equation (1) and (2), we get

$$\frac{C_1}{a}=-\frac{\mu I}{2\pi a} \quad \Rightarrow \quad C_1=-\frac{\mu I}{2\pi}$$

$$\therefore A_z=-\frac{\mu I}{2\pi}\ln\left(\frac{r}{a}\right)$$

In vector form, the vector magnetic potential is given as:

$$\vec{A}=-\frac{\mu I}{2\pi}\ln\left(\frac{r}{a}\right)\hat{a}_z$$

Proceeding in the same way as in equation (1), we get the magnetic induction as:

$$\vec{B}=\frac{\mu I}{2\pi r}\hat{a}_\phi$$

To summarise the results:

$$\vec{A}=\frac{\mu I}{4\pi}\left[1-\frac{r^2}{a^2}\right]\hat{a}_z \qquad r<a$$

$$=-\frac{\mu I}{2\pi}\ln\left(\frac{r}{a}\right)\hat{a}_z \qquad r>a$$

$$\vec{B}=\frac{\mu I r}{2\pi a^2}\hat{a}_\phi \qquad r<a$$

$$=\frac{\mu I}{2\pi r}\hat{a}_\phi \qquad r>a$$

Ques 16) Given the magnetic vector potential, $\vec{A}=-\frac{P^2}{4}\hat{a}_z$ Wb/m, calculate the total flux crossing the surface $\phi = \pi/2$, $1 \le \rho \le 2$ m, $0 \le z \le 5$m.

Ans: The magnetic flux density is,

$$\vec{B}=\nabla\times\vec{A}=-\frac{\partial A_z}{\partial\rho}\hat{a}_\phi=\frac{\rho}{2}\hat{a}_\phi$$

Differential surface is given as, $d\vec{S}=d\rho\, dz\,\hat{a}_\phi$

Hence, total flux crossing the given surface is given as:

$$\phi=\int_S \vec{B}\cdot d\vec{S}=\int_{z=0}^{5}\int_{\rho=1}^{2}\frac{\rho}{4}\hat{a}_\phi\cdot d\rho\, dz\,\hat{a}_\phi=\frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2}\rho\, d\rho\, dz$$

$$=\frac{1}{4}\left[\rho^2\right]_1^2\times 5=\frac{15}{4}\text{ Wb}$$

Module 4

Electric and Magnetic Field in Materials

ELECTRIC & MAGNETIC FIELD IN MATERIALS

Ques 1) Explain electric field in material and write properties of material.

Ans: Electric Field in Material
Electric fields can exist in free space, they exist in material media. Materials are broadly classified in terms of their electrical properties as conductors, semiconductors and insulators. Non-conducting materials are usually referred to as insulators or dielectrics.

A conductor is a material which contains movable electric charges. Metals such as copper aluminium are examples of conductors. In a Conductor the outer electrons of the atoms are loosely bound and free to move through the material. In conductors, the valence electrons are essentially free and strongly repel each other. Any external influence which moves one of them will cause a repulsion of other electrons which propagates through the conductor.

In an insulator the free electric charges are very few in number. Most solid materials are classified as insulators because they offer very large resistance to the flow of electric current. In insulators the outermost electrons are so tightly bound that there is essentially zero electron flow through them with ordinary voltages.

Properties of Material
The properties of semiconductors lie in between conductors and insulators. This is classified in terms of their conductivity:
1) High conductivity (>> 1) is referred to as a metal
2) Low conductivity (<< 1) is referred to as a dielectrics (or insulators)
3) The rest is semiconductor

Ques 2) Discuss the magnetic field in material.

Ans: Magnetic Field in Material
In magnetic terms, atoms and molecules inside matter resemble tiny current loops. If a piece of matter is situated in a magnetic field, the moment of magnetic forces partly aligns these loops, and we say that the substance is magnetised The magnetic field produced by the substance is due to these aligned current loops, known as Ampere's currents. A substance in the magnetic field can therefore be visualised as a large set of oriented elementary current loops situated in a vacuum. These oriented loops can be replaced by equivalent macroscopic currents situated in a vacuum, known as the magnetisation currents.

An elementary current loop is first characterised by a magnetic moment, m = I S, where I is the loop current and S its vector area. Next the magnetisation vector, M, is defined, as

$$M = \frac{(\sum m)_{in\,dv}}{dv} = Nm \ (A/m), \quad .....(1)$$

where, N is the number of Ampere's currents per unit volume.

The Ampere currents can be considered to be situated in a vacuum. Consequently, they can be incorporated in Ampere's law (which is valid for currents in a vacuum):

$$\oint_C B \cdot dl = \mu_0 \left(\int_S J \cdot dS + \oint_C M \cdot dl \right) \quad .....(2)$$

or

$$\oint_C H \cdot dl = \int_S J \cdot dS, \quad .....(3)$$

where,

$$H = B/\mu_0 - M \ (A/m) \quad .....(4)$$

is known as the **magnetic field intensity vector.**

Ques 3) Discuss polarization in dielectric medium.

Or

Explain the phenomenon of polarization and also derive a relation between polarisation vector (P), displacement (D) and electric field (E).

Ans: Polarization in Dielectric
When an electric field is applied across a dielectric material then the dielectric material becomes polarized. This mechanism is called **dielectric polarisation.**

If a material contains polar molecules, they will generally be in random orientations when no electric field is applied. An applied electric field will polarise the material by orienting the dipole moments of polar molecules.

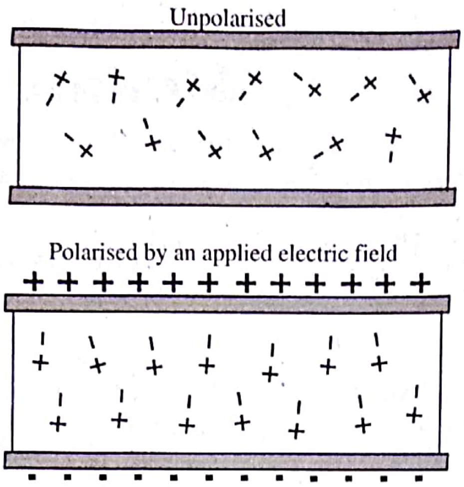


Figure 4.1

This decreases the effective electric field between the plates and will increase the capacitance of the parallel plate structure. The dielectric must be a polarised by an applied electric field, good electric insulator so as to minimise any DC leakage current through a capacitor.

The presence of the dielectric decreases the electric field produced by a given charge density

$$E_{effective} = E - E_{polarization} = \frac{\sigma}{k\varepsilon_0}$$

The factor k by which the effective field is decreased by the polarisation of the dielectric is called the **dielectric constant of the material.**

The main **difference between a conductor and a dielectric** is the availability of free electrons in the outermost atomic shells to conduct current.

Carriers in a dielectric are bounded by finite forces and as such, electric displacement occurs when external forces are applied. Such displacements are produced when an applied electric field, E, creates dipoles within the media that polarise it. Polarised media are evaluated by summing the original charge distribution and the dipole moment induced. One may also define the polarisation, P, of the material as the dipole moment per unit volume is,

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n} q_k \vec{d}_k}{\Delta v} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n} \vec{p}}{\Delta v}$$

Two types of dielectrics exist in nature:

1) **Non-Polar:** Non-polar dielectrics do not possess dipole moments until a strong electric field is applied.
2) **Polar:** Polar dielectrics such as water possess permanent dipole moments that further align (if possible) in the presence of an external field.

Relation between Polarisation Vector (P), Displacement (D) and Electric Field (E)

The effect on dielectric placed in an external electric field E_0 and there will be electric field due to polarised charges, this field is called electric field due to polarisation (E_p).

That is:

$$E = E_0 - E_p \qquad .....(1)$$

Polarisation vector, P is equal to the bound charge per unit area or equal to the surface density of bound charges (because surface charge density is charge per unit area),

Thus, $P = q_b/A = \sigma_p \qquad .....(2)$

Where q_b is bound charge and σ_p is surface density of bound charges.

P is also defined as the electric dipole moment of material per unit volume.

$$P = np$$

Where n is number of molecules per unit volume.

Displacement vector, D is equal to the free charge per unit area or equal to the surface density of free charges,

Thus $D = q/A = \sigma \qquad .....(3)$

Where q is free charge and σ is surface density of free charges.

As for parallel plate capacitor (already derived in earlier articles):

$$E = \sigma_p/\varepsilon_0 \qquad .....(4)$$
$$E_p = \sigma_p/\varepsilon_0 \qquad .....(5)$$

By substituting equations (4) and (5) in equation (1), we get

$$E = \sigma/\varepsilon_0 - \sigma_p/\varepsilon_0$$

Or $\varepsilon_0 E = \sigma - \sigma_0$

By putting equations (2) and (3) in above equation, we get

$$\varepsilon_0 E = D - P$$

Or $D = \varepsilon_0 E + P$

This is the relation between D, E and P.

Ques 4) Explain briefly the different types of polarisation in dielectrics.

Or

What is molecular polarisability? Explain electronic polarisability.

Or

Write different mechanisms of polarisation in a dielectric.

Ans: Mechanisms of Polarisation in a Dielectric/ Types of Polarisation

There are four different mechanisms by which dielectric polarisation occur:

1) **Electronic Polarisation:** When a dielectric is placed in on electric field, there is a displacement of electron

cloud relative to nuclei in the atom forming the molecule of dielectric. It causes an induced dipole moment in the molecule. This phenomenon is called electronic polarisation.

It results from the displacement of the centre of the negative charged electrons cloud relative to the positive nucleus of an atom by the electric field. This shifting of electrons could result in a dipole moment. Dipole moment is defined as the product of the charge and the shift distance.

$$P = qd$$

Also $\vec{P}$ is directly proportional to field strength

$$\vec{P} \propto \vec{E}$$

$$\vec{P} = \alpha \vec{E}$$

α (Alfa) is proportionality constant and known as electric polarisability and it is independent of temperature.

Mono atomic gas exhibits only this kind of polarisation.

$$\vec{P_e} = n\,\vec{p} = n\alpha_e\,\vec{E}$$

The contributions of $\vec{P}$ to the dielectric constant may be obtained as follows,

$$k = 1 + \chi = 1 + \frac{P}{\varepsilon_0 E} = 1 + \frac{n\alpha E}{\varepsilon_0 E}$$

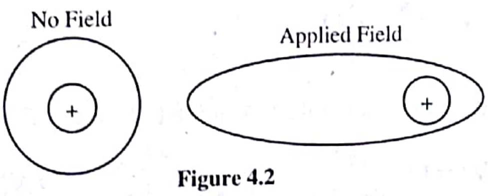


Figure 4.2

$$\varepsilon_r = 1 + \chi = 1 + \frac{P_e}{\varepsilon_0 E} = 1 + \frac{\alpha_e En}{\varepsilon_0 E}$$

$$\varepsilon_r = 1 + \frac{n\alpha}{\varepsilon_0} \text{ or } K = 1 + \frac{n\alpha_e}{\varepsilon_0}$$

The above expression indicates the dielectric constant due to electronic polarisation alone and thus gives the dielectric constant of non-polar gaseous dielectric.

2) **Ionic Polarisation:** Ionic Polarisation results from the separation of +ve and –ve ion air molecule held together by ionic bonds. When an electric field is applied to such a molecule their positive and negative ion are displaces further in opposite direction and their inter-ionic separation increases until bounding force to stop the process and thus increasing the dipole moment. The induced dipole moment due to ionic polarisation will be

$$\vec{P_i} = \alpha iE$$

Such molecules have a built up permanent dipole moment which exist even in the absence of electric field.

(a) E = 0 **(b) When electric field E Exists Separation Increases**

Figure 4.3

3) **Orientational or Dipolar Polarisation:** When the polar molecules are subjected to an electric field, the randomly distributed dipole field orient themselves with the applied field. This tendency of orientation of dipole along the field direction is called **orientational** or **dipolar polarisation.**

The orientation polarisation is temperature and frequency dependent. It take relatively large time to align along the applied field.

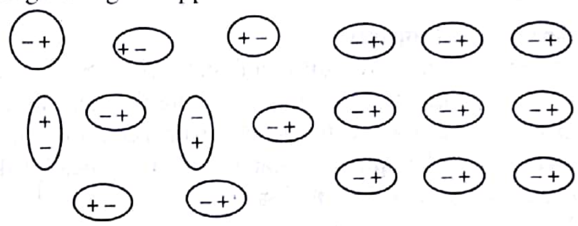


Figure 4.4

If n is the number of molecule then orientational polarisation.

$$P_0 = n\alpha_0 E$$

4) **Space-Charge Polarisation:** This type of polarisation occurs due to the accumulation of charge at the electrode or at the inter phase in a multiphase material, the ions diffuse over appreciable distance in response to the applied field, giving rise to a redistributions of charge in the dielectric medium. This type is also known as interfacial polarisation.

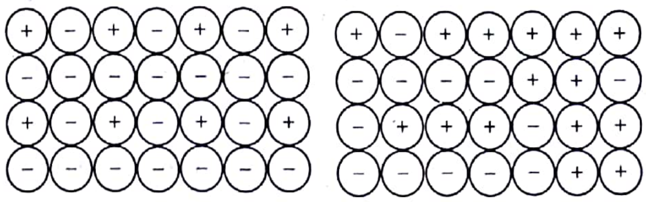


Figure 4.5

Now the total polarisation p of a multiphase material is equal to the sum of differential type of polarisation.

$$P = P_e + P_i + P_0 + P_s$$

Ques 5) Discuss the nature of dielectric materials and dielectric mechanism.

Ans: Dielectrics Materials

Dielectric is a non-conducting substance, i.e., an insulator. An ideal dielectric material is one which has no free charges. Although "dielectric" and "insulator" are generally considered synonymous, the term "dielectric" is more often used when considered the effect of alternating electric fields on the substance while "insulator" is more often used when the material is being used to withstand a high electric field.

Dielectrics are not a narrow class of so-called insulators, but the broad expanse of non-metals considered from the standpoint of their interaction with electric, magnetic, or electromagnetic fields. Thus we are concerned with gases as well as with liquids and solids, and with the storage of electric and magnetic energy as well as its dissipation.

Dielectric is the study of dielectric materials and involves physical models to describe how an electric field behaves inside a material. It is characterised by how an electric field interacts with an atom and is therefore possible to approach from either a classical interpretation or a quantum one. Many phenomena in electronics, solid state and optical physics can be described using the underlying assumptions of the dielectric model. This can mean that the same mathematical objects can go by many different names.

Dielectric Mechanisms

There are a number of different dielectric mechanisms, connected to the way a studied medium reacts to the applied field. Each dielectric mechanism is centred around its characteristic frequency, which is the reciprocal of the characteristic time of the process.

In general, dielectric mechanisms can be divided into relaxation and resonance processes. The most common, starting from high frequencies, are:

1) **Electronic Polarisation:** This resonant process occurs in a neutral atom when the electric field displaces the electron density relative to the nucleus it surrounds. This displacement occurs due to the equilibrium between restoration and electric forces.

2) **Atomic Polarisation:** This is observed when the electronic cloud is deformed under the force of the applied field, so that the negative and positive charges are formed. This is a resonant process.

Ques 6) Determine the energy density in electrostatic fields.

Or

Derive the expression for energy stored in an electric field.

Ans: Energy Density in Electrostatic Fields /Energy Stored in an Electrostatic Field

Energy stored in an electrostatic field is given by,

$$W_E = \frac{1}{2}\varepsilon_0 E^2 \text{Joules/m}^3$$

When a positive charge is brought from a distance of infinity to a point in a field of another positive charge work is done by an external source. Energy spent in doing so represents the potential energy.

If the external source is removed the charge that is brought moves back, acquiring kinetic energy of its own and it is capable of doing some work.

If V is the potential at a point due to some fixed charge,

Work done = potential energy

That is, $W_E = QV$

Where, Q is the charge brought by an external source,

V is the potential at the point due to a fixed charge.

Let us consider two charges Q_1 and Q_2, separated by a distance of infinity. If Q_1 is fixed, work done on bringing Q_2 towards Q_1 given by,

$$W_2 = Q_2V_2^1$$

Where

V_2^1 = potential of Q_1 at Q_2

Similarly, consider another charge, Q_3 which is at infinity from Q_1 and Q_2. Work done in bringing Q_3 towards Q_1 and Q_2 is given by,

$$W_3 = Q_3V_3^1 + Q_3V_3^2$$

This is because there exists force due to Q_1 and Q_2 after Q_2 is brought to Q_1.

In the above equation, V_3^1 and V_3^2 are the potentials at Q_3 due to Q_1 and Q_2 respectively. Therefore, total work done in bringing Q_2 and Q_3 is,

$$W_e = W_2 + W_3$$

In a similar fashion, consider n charges. Then we have

$$W_t = Q_1V_2^1 + Q_3V_3^1 + Q_3V_3^2) + (Q_4V_4^1 + Q_4V_4^2 + Q_4V_4^3) + (Q_nV_n^1 + Q_nV_n^2 + \ldots + Q_n V_n^{n-1})$$

that is, $$W_t = \sum_{i=2}^{n}\sum_{j=1}^{i=1} Q_i V_i^j$$

Where, V_i^j is the potential of Q_j at the location of Q_i.

And $$Q_jV_i^j = Q_i\frac{Q_j}{4\pi\epsilon_0 R_{ij}} = Q_j\frac{Q_i}{4\pi\epsilon_0 R_{ji}} = Q_jV_j^i$$

W_t is written as,

$$W_t = Q_1V_1^2 + Q_1V_1^3 + Q_2V_2^3) + (Q_1V_1^4 + Q_2V_2^4 + Q_3V_3^4) \ldots + (Q_1V_1^n + Q_2V_2^n + \ldots + Q_{n-1}V^n_{n-1})$$

Adding the above two equations and simplifying, we get,

$$2W_t = Q_1(V_1^2 + V_1^3 + V_1^4 + \ldots) + Q_2(V_2^1 + V_2^3 + V_2^4 + \ldots) + Q_3(V_3^1 + V_3^2 + V_3^4 + \ldots) + \ldots$$

= Q_1 × (potential at Q_1 due to all other charges) + Q_2 × (potential at Q_2 due to all other charges) + Q_n (potential at Q_n due to all other charges)

$$= Q_1V_1 + Q_2V_2 + \ldots Q_nV_n$$

$$2W_t = \sum_{i=1}^{n} Q_i V_i$$

Or $$W_t = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i$$

This equation represents the potential energy stored in a system of n point charges,

If $Q_i = \int_v \rho_v dv$

$$W_E = \frac{1}{2}\int_v \rho_v V dv$$

But, $\nabla.D = \rho_v$ or $\nabla.E = \rho_v / \varepsilon_0$

$$W_t = \frac{1}{2}\int_v \varepsilon_0(\nabla.E)Vdv$$

From standard vector identity,

$(\nabla.E)V = \nabla.VE - E.\nabla V$

This becomes, $W_t = \frac{1}{2}\varepsilon_0\int_v(\nabla.VE - E.\nabla V)dv$

$$= \frac{1}{2}\varepsilon_0\int_v(\nabla.VE\,dv + \frac{1}{2}\varepsilon_0\int_v E.E\,dv$$

Applying divergence theorem, first term on the right hand side can be written as,

$$\int_v \nabla.VE\,dv = \int V\ E.dS$$

Complete space — Surface bounding the space

However, viewing from a surface bounding complete space, the charge distribution of finite volume appears as a point charge, say Q. We know that

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}a_r$$

And $V = \frac{Q}{4\pi\varepsilon_0 r}$

From the expressions of E and V_r we get

$$\int_{\text{Surface boundary the space}} V\,E.dSa_n$$

$$= \underset{r\to\infty}{Lt}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q}{4\pi\varepsilon_0 r}\times\frac{Q}{4\pi\varepsilon_0 r^2}a_r.r^2 \sin\theta d\theta d\phi a_n$$

[as E is a_r-directed, dS is a_n-directed and ds = $r^2 \sin\theta\, d\theta\, d\phi$]

$$= \underset{r\to\infty}{Lt}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q^2}{4\pi\varepsilon_0 r}\sin\theta\, d\theta\, d\phi = 0$$

Hence W_1 becomes, $W_t = \frac{1}{2}\varepsilon_0 \int_{\text{Complete space}} E.E\,dv$

Or, $W_t = \int_{\text{Complete space}} \frac{1}{2}\varepsilon_0 E^2\,dv$ Joules

The above expression is the total **energy stored in an electrostatic field.**

Therefore, the integrand represents energy density, that is, **energy density in electrostatic field** is given by,

$$W_E = \frac{1}{2}\varepsilon_0 E^2 \text{ Joules/m}^3$$

Ques 7) There point charge –1nC, 4nC, 3nC, are located at (0, 0, 0) (0, 0, 1) (1, 0, 0) find energy in the system.

Ans: Using the energy formula,

$$w = \frac{1}{2}\sum_{K=1}^{3}Q_k V_K = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

$$\Rightarrow \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(\sqrt{2})}\right]$$

$$+\frac{Q_3}{2}\left[\frac{Q_1}{4\pi\varepsilon_0(1)} + \frac{Q_2}{4\pi\epsilon_0(\sqrt{2})}\right]$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$\Rightarrow 9\left(\frac{12}{\sqrt{2}} - 7\right)nJ = 13.37nJ$$

Ques 8) Calculate E at P (1, 1, 1) in free space caused by four identical 3-nC point chargers located at P_1 = (1, 1, 0) P_2 = (–1, 1, 0), P_3 = (–1, –1, 0) and P_4 = (1, –1, 0).

Ans: ∵ we find that,

$r = a_x + a_y + a_z$

$r_1 = a_x + a_y$

and thus, $r - r_1 = a_z$

The magnitudes are:

$|r - r_1| = 1$

$|r - r_2| = \sqrt{5}$

$|r - r_3| = 3$

$|r - r_4| = \sqrt{5}$

$$\because\quad E = \frac{1}{4\pi\epsilon_0}\sum_{K=1}^{N}\frac{Q_k(r - r_k)}{|r - r_k|^3}$$

$$\text{Or}\quad E = \frac{Q_1(r - r_1)}{4\pi\epsilon_0|r - r_1|^3} + \frac{Q_2(r - r_2)}{4\pi\epsilon_0|r - r_2|^3} + ...$$

$(\because Q = Q_1 = Q_2 = Q_3 ....)$

$$\text{Or } E = \frac{Q_1}{4\pi\epsilon_0}\left[\frac{r - r_1}{|r - r_1|^3} + \frac{r - r_2}{|r - r_2|^3} + \frac{r - r_3}{|r - r_3|^3} + \frac{r - r_4}{|r - r_4|^3} + ...\right]$$

Put the value of all the component which we find above.

$$E = \frac{3\times10^{-9}}{4\pi\times8.854\times10^{-12}}\left[\begin{array}{l}\frac{a_z}{1}.\frac{1}{1^2} + \frac{2a_x + a_z}{\sqrt{5}}.\frac{1}{(\sqrt{5})^2} \\ + \frac{2a_x + a_y + a_z}{3}.\frac{1}{3^2} + \frac{2a_y + a_z}{\sqrt{5}}.\frac{1}{(\sqrt{5})^2}\end{array}\right]$$

$$E = 26.96\left[a_z \cdot 1 + \frac{2a_x + a_z}{\sqrt{5}} \cdot \frac{1}{25} + \frac{2a_x + 2a_y + a_z}{3} \cdot \frac{1}{9} + \frac{2a_y + a_z}{\sqrt{5}} \cdot \frac{1}{25}\right]$$

On solving we get;

$E = 6.82\, a_x + 6.82\, a_y + 32.8 a_z$ V/m

Ques 9) Determine the boundary conditions of electric field from Maxwell's laws.

Or

Derive the boundary conditions between dielectric of conductor and between two conductors.

Or

Explain the tangential and normal boundary conditions between two dielectrics for static electric fields.

Ans: Boundary Condition of Electric Field

When an electric field passes from one medium to other medium, it is important to study the conditions at the boundary between ,two media. Depending on nature of medium, there are two situations of the boundary conditions:

1) **Boundary Conditions between Dielectric of Conductor:** Now consider a boundary b/w dielectric and conductor as shown **figure 4.6**:

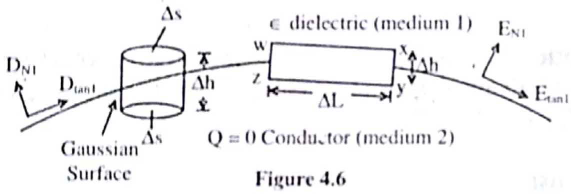


Figure 4.6

Inside a conductor external charges is zero because due to force these tends towards surface.

Now for Rectangle wxyz.

From Maxwell equation $\oint E \cdot d\ell = 0$.

$$\int_{w,x} E \cdot d\ell + \int_{x-surface} E \cdot d\ell + \int_{surface-y} E \cdot d\ell + \int_{yz} E \cdot d\ell + \int_{z-surface} E \cdot d\ell + \int_{surface-w} E \cdot d\ell = 0$$

$$E_{tan_1} \Delta L - E_{N_1} \cdot \frac{\Delta h}{2} + 0 + 0 + 0 + E_{N_1} \cdot \frac{\Delta h}{2} = 0$$

$E_{tan1} \cdot \Delta L = 0$

$\Rightarrow E_{tan1} = 0 \Rightarrow D_{tan1} = 0$ {since $D = \varepsilon E$}

i.e., tangential component of electric field intensity is zero.

Now consider cylinder, apply Gauss law.

$$\oint D \cdot ds = Q$$

$$\int_{top} D \cdot ds + \int_{bottom} D \cdot ds + \int_{lateral} D \cdot ds = Q$$

$$D_{N_1} \cdot \Delta s + 0 + 0 = Q = \rho_s \cdot \Delta s$$

$$D_{N_1} = \rho_s$$

Or $E_{N_1} = \dfrac{\rho_s}{\varepsilon}$

i.e., a normal component of flux density is equal to surface charge density.

2) **Boundary Conditions between Two Dielectric:** Consider a boundary between two dielectric as shown **figure 4.7**:

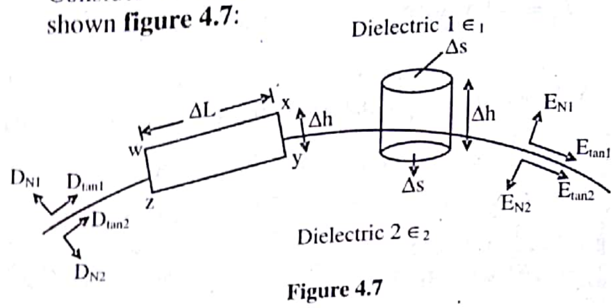


Figure 4.7

For Rectangle wxyz

From Maxwell equation $\oint E \cdot d\ell = 0$

$$\int_{w,x} E \cdot d\ell + \int_{x-surface} E \cdot d\ell + \int_{surface-y} E \cdot d\ell + \int_{yz} E \cdot d\ell + \int_{z-surface} E \cdot d\ell + \int_{surface-w} E \cdot d\ell = 0$$

$$\Rightarrow E_{tan_1} \Delta L + (-E_{N_1}) \cdot \frac{\Delta h}{2} - E_{N_2} \cdot \frac{\Delta h}{2} - E_{tan_2} \cdot \Delta L + E_{N_2} \cdot \frac{\Delta h}{2} + E_{N_1} \cdot \frac{\Delta h}{2} = 0$$

$\Delta h \approx$ very small so $\dfrac{\Delta h}{2} \rightarrow$ negligible.

$\Rightarrow (E_{tan1} - E_{tan2}) \cdot \Delta L = 0$

$$\Rightarrow E_{tan1} = E_{tan2} \text{ or } \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_1 \varepsilon_0}{\varepsilon_2 \varepsilon_0} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

i.e., tangential component of electric field intensity are equal for dielectric boundary.

For Cylinder

From Gauss Law $\oint D.ds = Q$

$$\int_{top} D \cdot ds + \int_{bottom} D \cdot ds + \int_{lateral} D \cdot ds = Q$$

$$D_{N_1} \cdot \Delta s - D_{N_2} \cdot \Delta s + 0 = Q$$

$$\Rightarrow (D_{N_1} - D_{N_2}) \Delta s = Q = \rho_s \cdot \Delta s$$

$$\Rightarrow D_{N_1} - D_{N_2} = \rho_s$$

For charge free boundary $\rho_s = 0$ than,

$$D_{N_1} = D_{N_2}$$

$$\Rightarrow \frac{E_{N_1}}{E_{N_2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}}$$

i.e., normal components of electric field intensity are inversely proportional to the relative permittivities of the two media.

If consider two different valves of medium i.e., D_1 and D_2 for two medium.

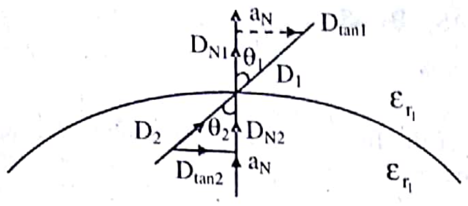


Figure 4.8

If θ_1 and θ_2 are the angle between D_{N_1}, D_1 and D_{N_2}, D_2.

As from boundary conditions, $D_{N_1} = D_{N_2}$ and $E_{tan1} = E_{tan2}$

$$D_{N_1} = D_1\cos\theta_1$$

And $D_{N_2} = D_2\cos\theta_2$

$$D_{tan1} = D_1\sin\theta_1$$

and $D_{tan2} = D_2\sin\theta_2$

$$\Rightarrow \frac{D_{tan1}}{D_{tan1}} = \frac{D_1\sin\theta_1}{D_2\sin\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}$$

$$\Rightarrow \varepsilon_{r_1} D_2\sin\theta_2 = \varepsilon_{r_2} D_1\sin\theta_1 \quad .....(1)$$

and $D_1\cos\theta_1 = D_2\cos\theta_2$ .....(2)

From (1) and (2) $\frac{D_1\sin\theta_1}{D_1\cos\theta_1}\varepsilon_{r_2} = \frac{D_2\sin\theta_2}{D_2\cos\theta_2}\varepsilon_{r_1}$

$$\Rightarrow \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}}$$

Ques 10) A potential field is given as $V = 100\, e^{-5x} \sin 3y \cos 4zV$. Let point (0.1, π/12, π/24) be located at a conductor free space boundary. At point P, find the magnitudes of,

1) V **2) $\overline{E}$**
3) E_t **4) E_N**
5) $\overline{D}$ **6) D_N**
7) ρ_s

Ans:

1) At P, $x = 0.1,\ y = \frac{\pi}{12},\ z = \frac{\pi}{24}$

$$\therefore\ V = 100\, e^{-0.5}\sin\frac{3\pi}{12}\cos\frac{4\pi}{24} = 37.1422V$$

.....Use radian mode

2) $\overline{E} = -\nabla V = -\left(\frac{\partial V_x}{\partial x}\bar{a}_x + \frac{\partial V_y}{\partial y}\bar{a}_y + \frac{\partial V_z}{\partial z}\bar{a}_z\right)$

$$= -100[-5e^{-5x}\sin 3y\cos 4z\,\bar{a}_x + e^{-5x}(3)(\cos 3y)(\cos 4z)\,\bar{a}_y + e^{-5x}(\sin 3y)(4)(-\sin 4z)\bar{a}_z]$$

At P, $\overline{E} = -100[-1.857\bar{a}_x + 1.114\bar{a}_y - 0.85776\bar{a}_z]$
$= +185.7\bar{a}_x - 111.4\bar{a}_y + 85.776\bar{a}_z$ V/m

$\therefore\ |\overline{E}| = 232.9206$ V/m

3) $E_t = 0$ V/m as P is on the boundary

4) $E_N = |\overline{E}| = 232.9206$ V/m

5) $\overline{D} = \varepsilon_0\overline{E} = 8.854\times10^{-12}$ $[185.7\bar{a}_x - 111.4\bar{a}_y + 85.776\bar{a}_z]$
$= 1.588\bar{a}_x - 0.9529\bar{a}_y + 0.7337\bar{a}_z$ nC/m²

$\therefore\ |\overline{D}| = 1.992$ C/m²

6) $D_N = |\overline{D}| = 1.992$ nC/m²

7) $D_N = \rho_s = 1.992$ nC/m²

Ques 11) Given that $\overline{E}_1 = 2\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z$ V/m at the charge free dielectric interface as shown in the figure 4.9. Find $\overline{D}_2$ and the angle θ_1, θ_2.

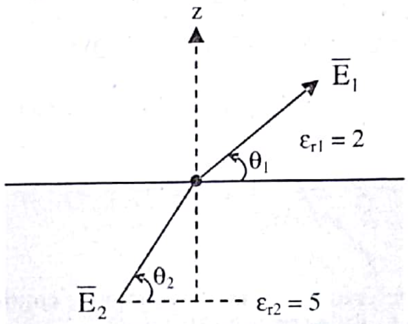


Figure 4.9

Ans: As shown, z axis is normal to the surface. So part of $\overline{E}_1$ which is in the direction of $\bar{a}_z$ is normal component of $\overline{E}_1$.

$\therefore\ \overline{E}_{N1} = 5\bar{a}_z$ V/m

And $\overline{E}_1 = \overline{E}_{N1} + \overline{E}_{tan1}$

$\because\ \overline{E}_{tan1} = \overline{E}_1 - \overline{E}_{N1} = 2\bar{a}_x - 3\bar{a}_y$ V/m .....(1)

At the boundary of perfect dielectrics,

$\overline{E}_{tan1} = \overline{E}_{tan2} = 2\bar{a}_x - 3\bar{a}_y$ V/m ....(2)

Now, $\overline{D}_{tan2} = \varepsilon_0\overline{E}_{tan2} = \varepsilon_0\varepsilon_{r2}\overline{E}_{tan2}$ .....(3)

And, $\overline{D}_{N1} = \varepsilon_1\overline{E}_{N1} = \varepsilon_0\varepsilon_{r1}\overline{E}_{N1}$ .....(4)

But, $\overline{D}_{N2} = \overline{D}_{N1} = \varepsilon_0\varepsilon_{r1}\overline{E}_{N1}$ .....(5)

And $\overline{D}_2 = \overline{D}_{N2} + \overline{D}_{tan2} = \varepsilon_0\varepsilon_{r1}\overline{E}_{N1} + \varepsilon_0\varepsilon_{r2}\overline{E}_{tan2}$
$= \varepsilon_0[5(2\bar{a}_x - 3\bar{a}_y) + 2(5\bar{a}_z)]$
$= 8.854\times10^{-12}[10\bar{a}_x - 15\bar{a}_y + 10\bar{a}_z]$

$\therefore\ \overline{D}_2 = 88.54\bar{a}_x - 132.81\bar{a}_y + 88.54\bar{a}_z$ pC/m³

As D_{N1}, E_{N1}, are in same direction and D_1, E_1, are in same direction,

$D_{N1} = D_1 \cos\theta_1'$, i.e., $E_{N1} = E_1 \cos\theta_1'$

Where θ_1' is angle measured w.r.t normal.

$$|E_{N1}| = 5 \text{ and } |E_1| = \sqrt{(2)^2 + (-3)^2 + (5)^2} = 6.1644$$

$$\therefore \quad \cos\theta_1' = \frac{E_{N1}}{E_1} = \frac{5}{6.1644}$$

$$\therefore \quad \theta_1' = 35.795°$$

This θ_1' is angle made by $\bar{E}_1$ with the normal while θ_1 is shown with respect to horizontal.

$$\therefore \quad \theta_1 = 90 - \theta_1' = 90 - 35.795 = 54.205°$$

Similarly if θ_2' is angle made by $\bar{E}_2$ with the normal then,

$$\cos\theta_2' = \frac{E_{N2}}{E_2} = \frac{D_{N2}}{D_2} = \frac{D_{N1}}{D_2} \quad (D_{N2} = D_{N1})$$

$$= \frac{\varepsilon_0 \varepsilon_{r1} |\bar{E}_{N1}|}{|\bar{D}_2|}$$

$$= \frac{\varepsilon_0 \times 2 \times 5}{\varepsilon_0 \times \sqrt{10^2 + (-15)^2 + (10)^2}}$$

$$= \frac{10}{20.6155} = 0.485$$

$$\therefore \quad \theta_2' = 60.982°$$

$$\therefore \quad \theta_2 = 90 - \theta_2' = 29.017°$$

Ques 12) Determine the boundary conditions of magnetic field from Maxwell's laws.

Ans: Boundary Condition of Magnetic Field

Magnetic boundary conditions are the conditions that a $\vec{B}$ or $\vec{H}$ (or $\vec{M}$) field must satisfy at the boundary between two different magnetic media.

To determine the conditions, we use Gauss' law of magnetostatics and Ampere's circuital law,

$$\oint_S \vec{B}.d\vec{S} = 0 \quad \text{and} \quad \oint_S \vec{H}.\vec{dl} = I_{enc}$$

We consider two different magnetic media 1 and 2, characterised by the permeabilities μ_1 and μ_2, respectively.

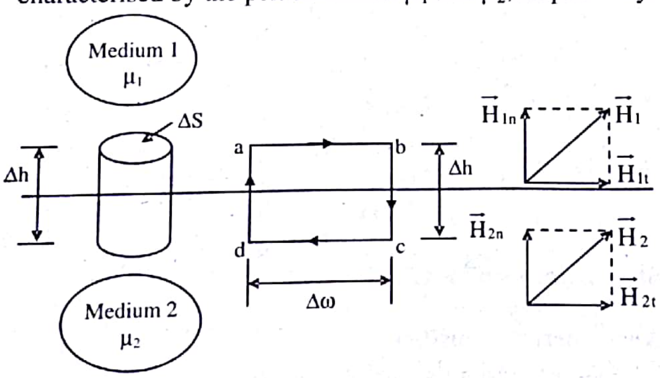


Figure 4.10: Magnetic Boundary Conditions

Apply Gauss's law to the pillbox (Gaussion surface), with $\Delta h \to 0$,

$$B_{1n}\Delta S - B_{2n}\Delta S = 0$$

$$B_{1n} = B_{2n} \quad .....(1)$$

In term of the field intensity, the boundary condition can be written as,

$$\mu_1 H_{1n} = \mu_2 H_{2n} \quad .....(2)$$

Thus, the normal component of $\vec{B}$ is continuous, but the normal component of $\vec{H}$ is discontinuous at the boundary surface.

Now, applying Ampere's circuital law, assuming that the boundary carries a surface current $\vec{K}$ whose component normal to the plane of the closed path abcda is K (A/m),

$$K\Delta\omega = H_{1t}\Delta\omega - H_{1n}\frac{\Delta h}{2} - H_{2n}\frac{\Delta h}{2} - H_{2t}\Delta\omega$$

$$+ H_{2n}\frac{\Delta h}{2} + H_{1n}\frac{\Delta h}{2}$$

$$\therefore \quad (H_{1t} - H_{2t}) = K \quad .....(3)$$

In terms of the flux density, we have,

$$\left(\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2}\right) = K \quad .....(4)$$

Thus, the tangential component of $\vec{H}$ is also discontinuous. The directions are specified by using the cross product as,

$$(\vec{H}_1 - \vec{H}_2) \times \bar{a}_{n21} = \vec{K}$$

$$\text{Or} \quad (\vec{H}_1 - \vec{H}_2) = \vec{K} \times \vec{a}_{n12} \quad .....(5)$$

Where $\vec{a}_{n21}$ is the unit vector normal to the boundary directed from the medium 2 to the medium 1.

If the media are not conductors then the boundary is free of current, i.e., k = 0; then

$$H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \text{ and } B_{1n} = B_{2n} \quad .....(6)$$

If the fields make an angle θ with the respective normal to the interface then we can combine the boundary conditions as,

$$\frac{B_1 \sin\theta_1}{\mu_1} = \frac{B_2 \sin\theta_2}{\mu_2} \quad \text{and } B_1\cos\theta_1 = B_2\cos\theta_2$$

Combining,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \quad \text{or} \quad \mu_1 \cot\theta_1 = \mu_2 \cot\theta_2$$

Ques 13) There is a boundary at $z = 0$ between two magnetic materials. Permittivities are $\mu_1 = z\mu_0$ (H/m) for region 1 ($z > 0$) and $\mu_2 = 14\mu_0$ (H/m) for region 2 ($z < 0$). Surface current density $\vec{K}$ is $50\,\bar{a}_x$ (A/m) at the boundary ($z = 0$). There is a field $B_1 = 4\bar{a}_x - 6\bar{a}_y + 4\bar{a}_z$ (mT) in region 1. What is the flux density ($\bar{B}_2$) in region 2?

Ans: Given that $\bar{B}_1 = (4\bar{a}_x - 6\bar{a}_y + 4\bar{a}_z)$ (mT). It is shown in **figure 4.11**.

Normal component of $\bar{B}_1$ is obtained as

$$\bar{B}_{n1} = (\bar{B}_1 \cdot \bar{a}_n)\bar{a}_n$$

Or $= [(4\bar{a}_x - 6\bar{a}_y + 4\bar{a}_z) \cdot \bar{a}_z]\bar{a}_z$

Or $= 4\bar{a}_z \text{(mT)}$

And also,

$$\bar{B}_{n2} = \bar{B}_{n1} = 4\bar{a}_z \text{(mT)}$$

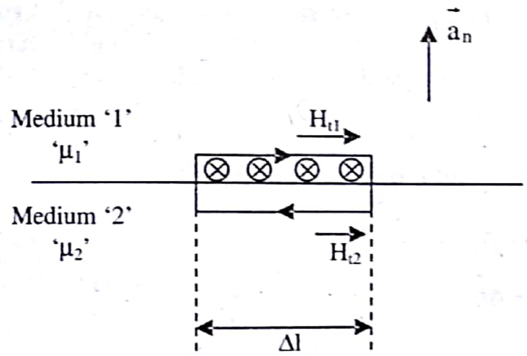


Figure 4.11: Two Magnetic Materials

Tangential component of $\bar{B}_1$ is obtained as

$$\bar{B}_{t_1} = (\bar{B}_1 - \bar{B}_{n_1})$$

$$= (4\bar{a}_x - 6\bar{a}_y)\text{(mT)}$$

Or $= (1592\bar{a}_x - 2388\bar{a}_y)$ (A/m)

Also, $\bar{H}_{t_1} = \dfrac{\bar{B}_{t_1}}{\mu_1}$

$$= \frac{(4\bar{a}_x - 6\bar{a}_y) \times 10^{-3}}{2 \times 4\pi \times 10^{-7}}$$

$$= (1592\bar{a}_x - 2388\bar{a}_y) \text{ (A/m)}$$

Further, $\bar{H}_{t_2} = \bar{H}_{t_1} - (\bar{K} \times \bar{a}_x)$

or $= [(1592\bar{a}_x - 2388\bar{a}_y) - (50\bar{a}_x \times \bar{a}_z)]$

$$= (1592\bar{a}_x - 2438\bar{a}_y)$$

Also, $\bar{B}_{t_2} = \mu_2 \bar{H}_{t_2}$

or $= 14 \times 4\pi \times 10^{-7}(1592\bar{a}_x - 2438\bar{a}_y)$

Finally, $\bar{B}_2$ can be obtained by using the relationship as given below:

$$\bar{B}_2 = \bar{B}_{n_2} + \bar{B}_{t_2} = 4\bar{a}_z + 14 \times 4\pi \times 10^{-7}(1592\bar{a}_x - 2438\bar{a}_y)$$

Ques 14) There are two homogenous, linear and isotropic, media with interface at $x = 0$. $x < 0$ describes medium 1 ($\mu_{r1} = 4$). $x > 0$ describes medium 2 ($\mu_{r2} = 10$). Magnetic field medium 1 is $(30\bar{a}_x - 80\bar{a}_y + 70\bar{a}_z)$ (A/m). Find the magnetic field in medium 2. Also, find the magnetic flux density in medium 1.

Ans: The magnetic field in medium 1 is obtained as

$$\bar{H}_1 = (30\bar{a}_x - 80\bar{a}_y + 70\bar{a}_z) \text{ (A/m)}$$

Further,

$$\bar{H}_1 = \bar{H}_{t_1} + \bar{H}_{n_1}$$

It is given that

$$\bar{H}_{t_1} = (-80\bar{a}_y + 70\bar{a}_z) \text{ and}$$

$$\bar{H}_{n_1} = 30\bar{a}_x$$

Applying boundary conditions, we get

$$\bar{H}_{t_1} = \bar{H}_{t_2}$$

or $\bar{H}_{t_2} = (-80\bar{a}_y + 70\bar{a}_z)$ (A/m)

Now applying boundary condition on $\bar{B}$, we get

$$\bar{B}_{n_1} = \bar{B}_{n_2}$$

or $\mu_1 \bar{H}_{n_1} = \mu_2 \bar{H}_{n_2}$

or $\bar{H}_{n_2} = \dfrac{\mu_1}{\mu_2}\bar{H}_{n_1} = \dfrac{\mu_{r_1}\mu_0}{\mu_{r_2}\mu_0}\bar{H}_{n_1}$

or $= \dfrac{4}{10} \times 30\bar{a}_x$

or $= 12\bar{a}_x$

$$\bar{H}_2 = \bar{H}_{t_2} + \bar{H}_{n_2}$$

or $= (+12\bar{a}_x - 80\bar{a}_y + 70\bar{a}_z)$ (A/m)

Magnetic flux density $\bar{B} = \mu_1 \bar{H}_1$ or $= \mu_{r_1}\mu_0 \bar{H}_1$

or $= 4 \times 4\pi \times 10^{-7}(30\bar{a}_x - 80\bar{a}_y + 70\bar{a}_z)$

or $= 16\pi(3\bar{a}_x - 8\bar{a}_y + 7\bar{a}_z)(\mu(\mu Wb^{-2})$

Ques 15) What do you mean by current density? Explain different types of current densities.

Or

Define conduction and convection currents.

Or

State point form of Ohm's law.

Ans: Current Densities

Current density is defined as the current at a given point through a unit normal area at that point. It is a vector and it has the unit of Ampere/ m^2. It is represented by J.

Types of Current Densities

There are two types of current densities:

1) **Convection Current Density:** Convection current occurs in materials having a few free electrons or ions, that is, in dielectrics (or insulators). It is produced due to the motion of electrons in an insulating medium, such as liquid, vacuum, etc., does not satisfy Ohm's law.

To derive a relation for convection current density, consider a filament shown in **figure 4.12** through which a charge Q of density ρ_v flows with a velocity u along the y-direction.

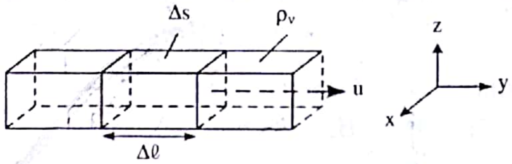


Figure 4.12: Convection Current in a Filament

As the charge is flowing in y-direction, let us take only the y-component of the velocity into account, which is $u_y \hat{a}_y$.. The element of charge ΔQ of density ρ_v enclosed in the filament of volume Δv is obtained as:

$$\Delta Q = \rho_v \Delta v$$

Thus, the element of the convection current through the filament is obtained as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \Delta v}{\Delta t}$$

The volume Δv in terms of surface area Δs can be represented as,

$$\Delta v = \Delta s \Delta \ell$$

Where, $\Delta\ell$ is the distance that the element of charge ΔQ has moved in the interval Δt.

Thus, $\Delta I = \rho_v \Delta s \frac{\Delta \ell}{\Delta t}$

$\Rightarrow \Delta I = \rho_v \Delta s u_y$

Where, $u_y = \frac{\Delta \ell}{\Delta t}$ is velocity of the moving electrons.

The expression for convection current density in y-direction (that is J_y) is obtained as:

$$J_y = \rho_v u_y$$

In general, the expression for the **convection current density** is given as:

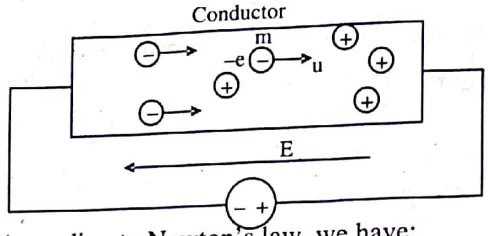
$$\vec{J} = \rho_v \vec{u}$$

2) **Conduction Current Density: Conduction current** occurs in materials in which there are a large number of free electrons that is in conductors. It is produced due to the motion of free electrons in a conductor on the application of an electric field. As it involves conductors, it satisfies Ohm's law.

When an external electric field $(\vec{E})$ is applied to a metallic conductor, the charge inside the conductor (electron) experiences a force, which is given as:

$$\vec{F} = -e\vec{E}$$

Where –e represents the negative charge on an electron.



According to Newton's law, we have;

$$\frac{mu}{\tau} = -eE$$

$$u = -\frac{e\tau}{m}E$$

Where (τ) is the average time interval between collisions, if there are (n) electrons per unit volume, the electron charge density is given by

$$\rho_v = -ne$$

Thus the **conduction current density** is;

$$J = \rho_v u = \frac{ne^2\tau}{m}E = \sigma E$$

$$J = \sigma E \quad \quad ....(8)$$

Here conductivity (σ) is measured in mhos per meter (℧/m).The equation (8) is called **point form of Ohm's law.** The unit of conductivity is also called Siemens per metre (S/m). For the metallic conductors the conductivity is constant over wide ranges of current density and electric field intensity. In all directions, metallic conductors have same properties hence called isotropic in nature. Such materials obey the Ohm's law very faithfully.

Ques 16) State continuity equation and also define relaxation time.

Ans: Continuity Equation

Continuity equation states that "if the net charge crossing a surface enclosing closed volume is not zero, then the charge density within the volume must charge with time in a manner that the rate of increase of charge within the volume equals the net rate of charge into the volume."

Let us assume that charge 'ρ' is a function of time. The transport of charge is responsible for current.

i.e., $I = \frac{dq}{dt}$

If ρ is volume charge density, which is given by

$$\rho = \frac{dq}{dV},$$

$$q = \int \rho dV$$

$$I = \frac{d}{dt}\int \rho dV \qquad \text{..... (1)}$$

Here, it has been considered that the current is as the extended in space of volume V closed by a surface S. The current density J is defined as net amount of charge crossing the unit area normal to the direction of charge flow of the surface in unit time.

$$I = \int \vec{J} \cdot d\vec{s} \qquad \text{..... (2)}$$

From equation (1) and (2), it becomes,

$$\int \vec{J} \cdot d\vec{s} = \frac{d}{dt}\int \rho dV$$

The current is flowing outward direction so current I should be negative and hence above relation becomes,

$$\int \vec{J} \cdot d\vec{s} = -\frac{d}{dt}\int \rho dV$$

Gauss divergence theorem,

$$\int \vec{J} \cdot d\vec{s} = \int (\nabla \cdot \vec{J}) d\vec{V}$$

$$\int (\nabla \cdot \vec{J}) d\vec{V} = -\frac{d}{dt}\rho \, dv$$

Or, $\int (\nabla \cdot \vec{J} + \frac{d\rho}{dt}) d\vec{V} = 0$

Since, the arbitrary volume $dV \neq o$,

$$\nabla \cdot \vec{J} + \frac{d\rho}{dt} = 0 \qquad \text{..... (3)}$$

This equation (3) is called the **equation of continuity.**

For stationary current ρ is constant. So, $\frac{d\rho}{dt} = 0$ Hence,

$$\nabla \cdot \vec{J} = 0$$

Relaxation Time

The relaxation time is the time it takes a charge placed in the interior of a material to drop by e^{-1} (=36.8%) of its initial value.

For good conductors T_r is approx. 2×10^{-19} s.

Assume that a charge is introduced at some point inside a given material. Since:

$$\vec{J} = \sigma\vec{E}$$

Substituting this value of $\vec{J}$ in equation (3), we get;

$$\nabla . \sigma\vec{E} = -\frac{\partial \rho_v}{\varepsilon} \qquad \text{.....(4)}$$

Now using Gauss's law,

$$\nabla . \vec{E} = \frac{\rho_v}{\varepsilon}$$

Thus, equation (4) can now be written as,

$$\frac{\sigma\rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t} \qquad \text{.....(5)}$$

$$\frac{\sigma\rho_v}{\varepsilon} + \frac{\partial \rho_v}{\partial t} = 0$$

Separating variables in the above equation, we get;

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t \qquad \text{.....(6)}$$

Integrating both sides of equation (6), we get;

$$\ell n \rho_v = -\frac{\sigma t}{\varepsilon} + \ell n \rho_{v\,int}$$

Where ρ_{vint} is the integration constant and represents the initial charge density at t = 0. The above equation can also be written as:

$$\ell n \frac{\rho_v}{\rho_{v\,int}} = -\frac{\sigma t}{\varepsilon}$$

$$\rho_v = \rho_{v\,int} e^{-\sigma t/\varepsilon}$$

Or $\rho_v = \rho_{v\,int} e^{-t/T_r}$ .....(7)

Where T_r represents the relaxation time. It is also known as re-arrangement time and is given as:

$$T_r = \frac{\varepsilon}{\sigma} \qquad \text{.....(8)}$$

Some important points regarding relaxation time are as follows:

1) For good conductors, the relaxation time is very short as the charged particles move very rapidly inside them and redistribute themselves at the surface quickly.

2) For good dielectrics, the relaxation time is very long as the charged particles are bound and thus not free to move in the dielectrics.

Ques 17) Explain the concept of displacement current and show how it led to the modification of Ampere's law.

Or

Explain the concept of displacement current.

Or

What is displacement current?

Ans: Displacement Current

Current in conductor produces magnetic field but a change in electric field in vacuum also produces magnetic field. So a changing electric field is equivalent to current and exists as long as the field is changing. This equivalent current produces the same magnetic effect as the ordinary current in a conductor. This equivalent current is known as **displacement current.**

Modification of Ampere's Law

We shall now consider Maxwell's curl equation for magnetic fields (Ampere's Law) for time-varying conditions.

For static fields EM fields, we have

$\nabla\times B = u_0 J$

Taking divergence of both sides

$\nabla.(\nabla\times B) = \nabla.\ \mu_0 J$

$0 = \nabla.(\mu_0 J)$

$= \mu_0 \nabla.J$ [Since, $\nabla.(\nabla\times A) = 0$]

$\therefore\ \nabla.J = 0$

This means that the current is always closed and there are no source and sink.

But the continuity equation is

$\nabla.J = -d\rho/dt$ for time varying fields

So this equation fails and it needs little modification. So Maxwell assume that,

$\nabla\times B = \mu_0 (J + J_d)$

Taking divergence both sides

$\nabla.(\nabla\times B) = \mu_0 [(\nabla.J) + (\nabla.J_d)]$

But $\nabla.(\nabla\times B) = 0$

$0 = \mu_0 (\nabla.J) + \mu_0(\nabla.J_d)$

$\nabla.J_{d} = \nabla.J$

By putting from continuity equation $\nabla.J = d\rho/dt$

$\nabla.J_d = d\rho/dt$ .....(1)

(By Maxwell first equation, $\nabla.D = \rho$ in equation (1))

$\nabla.J_d = d\ \nabla.D/dt$

$\nabla.J_d = \nabla.dD/dt$

Therefore, $J_d = dD/dt$

This additional term $J_d = dD/dt$ is displacement current density, A/m^2

The **Ampere's law** for time varying field takes the form, Putting in equation (4), we get

$\nabla\times B = \mu_0(J + dD/dt)$

$B = \mu_0 H$

$\nabla\times (\mu_0 H) = \mu_0(J + dD/dt)$

$\nabla\times H = (J + dD/dt)$

This equation is applicable for varying as well as steady currents.

Ques 18) Derive Maxwell's equation. Explain the physical significance of each Maxwell's equation.

Ans: The derivation **differential form** of Maxwell's Equations is discussed below:

1) **Derivative of First Equation:** "Maxwell first equation is nothing but the differential form of Gauss law of electrostatics."

$\text{Div } D = \Delta.D = \rho$

Let us consider a surface S bounding a volume V in a dielectric medium. In a dielectric medium total charge consists of free charge. If ρ is the charge density of free charge at a point in a small volume element dV, .

Then Gauss's law can be express as, "The total normal electrical induction over a closed surface is equal to – times of $1/\varepsilon_0$ total charge enclosed."

$$\oint_S E\hat{n}dS = \frac{q}{\varepsilon_0}$$

$$\oint_S E\hat{n}dS = \frac{1}{\varepsilon_0}\int_V \rho dV$$

Where,

ρ = charge per unit volume

V = volume enclosed by charge.

By Gauss transformation formula

$$\int_V \text{div } E\, dv = 1/\varepsilon_0 \int_V \rho\ dv$$

$$\int_S A_n\, ds = \int_V \nabla A\ dv$$

$\Delta.E = 1/\varepsilon_0\, \rho$

$\varepsilon_0\Delta.\ E = P$

$\Delta.\varepsilon_0\, E = P$

If ρ=0 then $\Delta.\ \varepsilon_0 E$=0

or $\Delta.D$=0 (since $D = \varepsilon_0 E$)

2) **Derivative of Second Equation:** The 2[nd] form of equation can be written as,

$\text{Div } B = \Delta.B = 0$

"It is nothing but the differential form of Gauss law of magnetostatics."

Since isolated magnetic poles and magnetic currents due to them have no significance. Therefore magnetic lines of force in general are either closed curves or go off to infinity. Consequently the number of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it. It means that the flux of magnetic induction B across any closed surface is always zero.

Gauss law of magnetostatics states that "Total normal magnetic induction over a closed surface is equal to zero."

i.e.: $\int_S B\, n\, ds = 0$

Applying Gauss transformation formula we get

$$\int_V \nabla.B\ dV = 0$$

The integrand should vanish for the surface boundary as the volume is arbitrary.

$\nabla.\ B = 0$

3) **Derivative of Third Law:** The 3rd law can be written as,

$$\nabla\times E=-\frac{\partial B}{\partial t}$$

"It is nothing but the differential form of Faraday's law of electromagnetic induction."

According to Faraday's law of electromagnetic induction, it is known that e.m.f. induced in a closed loop is defined as negative rate of change of magnetic flux, i.e., e = – dθ/dt

Where,
θ = Magnetic flux

Or $\phi=\int_s B\,n\,ds$

$\phi = B/A$

Where,
Any surface having loop as boundary

$$\int_\ell E\cdot d\ell=-d\phi dt \qquad .....(1)$$

Putting the value of ϕ in equation (1), we get

$$\int_\ell E\cdot d\ell=-d/dt\int_s B\cdot n\,ds$$

$$\int_\ell E.d\ell=\int_s dB/dt\cdot n\,ds$$

Applying Stoke's transformation formula on L.H.S,

$$\int_s \nabla\times E\hat{n}ds=\int_s -\frac{\partial B}{\partial t}n\,ds$$

Or $\int_s\left(\nabla\times E+\frac{\partial B}{\partial t}\right)n\,ds=0$

Further validity of the equation,

$$\nabla\times E=\frac{\partial B}{\partial t}=0$$

This is known as Maxwell's third equation.

4) **Derivative of Maxwell's Fourth Equation:** "This is nothing but differential form of modified Ampere circuital law." According to law, "The work done in carrying a unit magnetic pole once around closed arbitrary path linked with the current" is expressed by

$$\int_\ell B\,d\ell=\mu_o\times i$$

i = Current enclosed by the path

$$\int_\ell B\,d\ell=\mu_o\int_s n\,ds$$

Applying, Stoke's transformation formula in L.H.S.

$$\int s\nabla\times B\,n\,ds=\int_s \mu_o J n\,ds$$

$\int s(\nabla\times B-\mu_o J)\,n\,ds=0$

For the validity this equation

$\nabla\times B-\mu_0 J=0$

$\nabla\times B=\mu_0 J$

It is known as the fourth equation of Maxwell.

Taking divergence of both sides

$\nabla.(\nabla\times B)=\nabla.(\mu_0 J)$

$0=\nabla.(\mu_0 J)=\mu_0\nabla.J\ [\nabla.(\nabla\times A)=0]$

$\nabla.J=0$

This means that the current is always closed and there are no source and sink.

But the according to law of continuity,

$\nabla.J=-\partial p/\partial t$

So this equation fails and it need of little modification. So Maxwell assume that

$\nabla\times B=\mu_0(\nabla.J)+\mu_0(\nabla.J_d)$

$0=\mu_0(\nabla.J)+\mu_0(\nabla.J_d)$

By putting $\nabla.J_d=\partial p/\partial t$ .....(1)

$\nabla.J_d=\nabla.\partial D/\partial t$

$J_d=\partial D/\partial t$ (By Maxwell first equation, ∇.D = p in equation (1))

Putting in equation (1), we get;

$\nabla\times B=\mu_0(J+\partial D/\partial t)$

$B=\mu_0=H$

$\nabla\times(\mu_0=H)=\mu_0(J+\partial D/\partial t)$

$\nabla\times H=j\,\partial D/\partial t$

The Derivation of **Integral Forms** of Maxwell's Equations are discussed below:

1) **Maxwell First Equation:** Consider $\nabla\cdot\vec{D}=\rho$

$$\int_v(\nabla\cdot\vec{D})dv=\int_v\rho dv$$

$$\oint_S\vec{D}\cdot d\vec{s}=Q$$

2) **Maxwell Second Equation:** Consider $\partial\vec{B}=0$,

Take the volume integral of both sides of the divergence equation over a volume V and apply the divergence theorem:

$$\int_V(\partial\vec{B})dv=0$$

Results, $\oint_S\vec{B}\cdot\hat{n}ds=0$

3) **Maxwell Third Equation:** Consider $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ and integrate over an open

Or surface S with a contour C and apply Stoke's Theorem:

$$\int_S\nabla\times\vec{E}\cdot\hat{n}ds=-\int_S\frac{\partial\vec{B}}{\partial t}\cdot\hat{n}ds$$

4) **Maxwell Fourth Equation:** Now consider,

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

And, integrate over a surface S

$$\int_S \nabla \times \vec{H} \cdot \hat{n} ds = -\int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \hat{n} ds$$

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \hat{n} ds$$

$$\oint_C \vec{H} \cdot d\vec{\ell} = I + \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \hat{n} ds$$

Ques 19) Write the Maxwell's equation in differential and integral form from modified form of Ampere's circuital law, Faraday's law and Gauss law.

Or

Write the Maxwell's equation in differential and integral form for static EM field.

Or

Write the final form of maxwell equation.

Ans: Maxwell's equation from modified form of Ampere's circuital law, Faraday's law and Gauss law/ Maxwell's Equation for Static EM Fields/ final form of Maxwell's equation

The Maxwell's equations for static EM fields are shown in **table 4.1** below:

Table 4.1: Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot D = \rho_v$	$\oint_S D \cdot dS = \int_v \rho_v dV$	Gauss's Law
$\nabla \cdot B = 0$	$\oint_S B \cdot dS = 0$	Non-existence of magnetic monopole
$\nabla x E = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot d\ell = -\frac{\partial}{\partial t}\int_s B \cdot dS$	Faraday's Law
$\nabla x H = J + \frac{\partial D}{\partial t}$	$\oint_L H \cdot d\ell = \int_s J \cdot dS$	Ampere's Circuit Law

Module 5

Time Varying Electric and Magnetic Fields

TIME VARYING ELECTRIC AND MAGNETIC FIELDS

Ques 1) State and explain Faraday's law.

Ans: Faraday's Law

According to Faraday, a time varying magnetic field produces an induced voltage (called electromotive force or emf) in a closed circuit, which causes a flow of current.

The induced emf(V_{emf}) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is Faraday's law and can be expressed as:

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\psi}{dt}$$

Where, N is the number of turns in the circuit and ψ is the flux through each turn.

The negative sign shows that the induced voltage acts in such a way to oppose the flux producing in it. This is known as **Lenz's law.**

Ques 2) Explain transformer and motional electromotive forces.

Ans: Electromotive Forces

Faraday's law links electric and magnetic fields, for a circuit with a single (N = 1), faraday law becomes,

$$V_{emf} = -N\frac{d\psi}{dt} \quad .....(1)$$

In terms of E and B, equation (1) can be written as:

$$V_{emf} = \oint_L E \cdot d\ell = -\frac{d}{dt}\int_S B \cdot dS \quad .....(2)$$

Where, ψ has been replaced by $\int_S B \cdot dS$ and S is the surface area of the circuit bounded by the closed path L.

It is clear from equation (2) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that $d\ell$ and dS in equation (2) are in accordance with the right-hand rule as well as Stokes's theorem. This should be observed in **figure 5.1.**

The variation of flux with time as in equation (1) or equation (2) may be caused in three ways:

1) **Transformer EMF or Stationary Loop in Time-Varying B Field:** This is the case portrayed in **figure 5.1**where a stationary conducting loop is in a time varying magnetic B field. Equation (2) becomes

$$V_{emf} = \oint_L E \cdot d\ell = -\int_S \frac{\partial B}{\partial t} \cdot ds \quad .....(3)$$

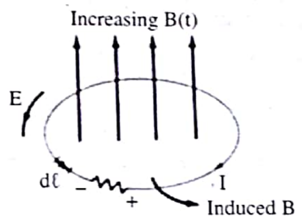


Figure 5.1: Induced EMF Due to a Stationary Loop in a Time Varying B Field

This emf induced by the time-varying current (producing the time-varying B field) in a stationary loop is often referred to as transformer emf in power analysis since it is due to transformer action. By applying Stokes's theorem to the middle term in equation (3), we obtain;

$$\int_S (\nabla \times E) \cdot dS \int_S \frac{\partial B}{\partial t} \cdot ds \quad ......(4)$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad .....(5)$$

This is one of the **Maxwell's equations** for time-varying fields. It shows that the time varying E field is not conservative ($\nabla \times E \neq 0$). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying electric field, e.g., is due to the energy from the time-varying magnetic field.

2) **Motional EMF or Moving Loop in Static B Field:** When a conducting loop is moving in a static B field, an emf is induced in the loop. We recall from equation (6) that the force on a charge moving with uniform velocity u in a magnetic field B is:

$$F_m = Qu \times B \quad ....(6)$$

We define the motional electric field E_m as:

$$E_m = \frac{F_m}{Q} = u \times B \qquad .....(7)$$

If we consider a conducting loop, moving with uniform velocity u as consisting of a large number of free electrons, the emf induced in the loop is:

$$V_{emf} = \oint_L E_m \cdot d\ell = \oint_L (u \times B) \cdot d\ell \qquad .....(8)$$

This type of emf is called motional emf or flux-cutting emf because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.

3) **Moving Loop in Time-Varying Field:** This is the general case in which a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present.

The total emf is obtained as,

$$V_{emf} = \oint_L E \cdot dI = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_L (u \times B) \cdot dI$$

Or above equations can be written as,

$$\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (u \times B)$$

Ques 3) What do you mean by power density of electromagnetic waves? Derive the average power transmitted by electromagnetic waves.

Ans: Power Density of Electromagnetic Waves

Power density is the time rate per unit of area at which electromagnetic energy flows through some medium. The quantity of energy is complexly related to the strengths of the E and the H fields. E × H has units of W/m^2. It represents the density of power carried by electromagnetic waves across the surface S. It is called as **Poynting Vector.**

Average Power Transmitted

The time-average Poynting vector indicates the average real power density of electromagnetic waves. It can be derived using time average of the complex poynting vector.

Let $E = Ex_0 e^{-\alpha\tau} \cos(wt - \beta\tau) a_x$

$$H = \frac{Ex_0}{2|\eta|} e^{-\alpha\tau} \cos(wt - \beta\tau - \theta_n) a_y$$

⇒

$$P = \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} 2\cos(wt - \beta\tau) \cos(wt - \beta\tau - \theta_n) a_z$$

$$P = \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \{\cos(A+B) + \cos(A+B)\} a_z$$

$$P = \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \begin{Bmatrix} \cos(wt - \beta\tau + wt - \beta\tau - \theta_n) \\ + \cos(wt - \beta\tau - wt + \beta\tau + \theta_n) \end{Bmatrix} a_z$$

$$P = \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \{\cos(2wt - 2\beta\tau - \theta_n) + \cos\theta_n\} a_z$$

$$Pav = \frac{1}{T} \int_0^T P(t) dt$$

$$= \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \left\{ \frac{1}{T} \int_0^T \cos(2wt - 2\beta\tau - \theta_n) + \frac{1}{T} \int_0^T \cos\theta_n dT \right\}$$

$$= \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \{0 + \cos\theta_n\}$$

$$= \frac{Ex_0^2}{2|\eta|} e^{-2\alpha\tau} \cos\theta_n$$

Ques 4) State and derive Poynting Vector theorem and also express it in complex form.

Or

What do you understand by Poynting Vector?

Or

What is the significance of the Poynting Vector in a static electromagnetic field?

Ans: Poynting Vector

Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to objects placed in their path. The rate of flow of energy in an electromagnetic wave is described by a vector S, called the Poynting vector, which is defined by the expression

$$\vec{S} = \vec{E} \times \vec{H} \left(= \frac{\vec{E} \times \vec{B}}{\mu} \right)$$

The magnitude of the Poynting vector represents the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation. Thus, the magnitude of the Poynting vector represents power per unit area. The direction of the vector is along the direction of wave propagation. The SI units of the Poynting vector are J/s.m^2 = W/m^2.

This theorem states that the cross product of electric field vector, E and magnetic field vector, H at any point is a measure of the rate of flow of electromagnetic energy per unit area at that point, that is,

$$P = E \times H$$

Here P → Poynting vector, the direction of P is perpendicular to E and H and in the direction of vector E × H.

Poynting Theorem

Poynting theorem states that net power flowing out of a given volume is equal to the time rate of decrease in energy stored within volume minus the ohmic loss.

The Poynting equation is given as,

$$\oint (E\times H)ds = -\int \sigma E^2 dv - \int \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dv$$

Power Transmited ⇓, Losses ⇓, Power Generated ⇓

i.e., total power leaving the volume = rate of decrease of stored electromagnetic energy – Ohmic power dissipated due to motion of charge

Proof: The energy density carried by the electromagnetic wave can be calculated using Maxwell's equations.

As
$$\text{div}\,\vec{D} = 0 \quad .....(1)$$

$$\text{div}\,\vec{B} = 0 \quad .....(2)$$

$$\text{Curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad .....(3)$$

And
$$\text{Curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad .....(4)$$

Taking scalar product of equation (3) with H and equation (4) with $\vec{E}$

i.e.
$$\vec{H}\,\text{curl}\,\vec{E} = -\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \quad .....(5)$$

And
$$\vec{E}\,\text{curl}\,\vec{H} = \vec{H}\,\vec{J} + \vec{E}.\frac{\partial \vec{D}}{\partial t} \quad .....(6)$$

Subtracting equation (6) from equation (5) i.e.

$$\vec{H}\,\text{curl}\,\vec{E} - \vec{E}.\text{curl}\,\vec{H} = -\vec{H}\frac{\partial \vec{B}}{\partial t} - \vec{E}\,\vec{J} - \vec{E}.\frac{\partial \vec{D}}{\partial t}$$

$$= -\left[\vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{E}.\frac{\partial \vec{D}}{\partial t}\right] - \vec{E}\,\vec{J}$$

as
$$\text{div}(\vec{A}\times\vec{B}) = \vec{B}\,\text{curl}\,\vec{A} - \vec{A}\,\text{curl}\,B$$

So
$$\text{div}\,(\vec{E}\times\vec{H} = -\left[\vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{E}.\frac{\partial \vec{D}}{\partial t}\right] - \vec{E}\,\vec{J} \quad .....(7)$$

But
$$\vec{B} = \mu\,\vec{H} \text{ and } \vec{D} = \varepsilon\,\vec{E}$$

So
$$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \vec{H}.\frac{\partial}{\partial t}(\mu\,\vec{H}) = \frac{1}{2}\mu\frac{\partial}{\partial t}(H^2)$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\vec{H}.\vec{B}\right]$$

And

$$\vec{E}\frac{\partial \vec{D}}{\partial t} = \vec{E}.\frac{\partial}{\partial t}(\varepsilon\vec{E}) = \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E^2) = \frac{\partial}{\partial t}\left[\frac{1}{2}\vec{E}.\vec{D}\right]$$

So from equation (7)

$$\text{div}\,(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}.\vec{B} + \vec{E}.\vec{D})\right] - \vec{E}.\vec{J}$$

Or
$$(\vec{E}.\vec{J}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}.\vec{B} + \vec{E}.\vec{D})\right] - \text{div}\,(\vec{E}\times\vec{H}) \quad ....(8)$$

Integrating equation (8) over a volume V enclosed by a surface S

$$\int_V \vec{E}.\vec{J}dV = \int_V \left|\frac{\partial}{\partial t}\left\{\frac{1}{2}(\vec{H}.\vec{B} + \vec{E}.\vec{D})\right\}\right| dV - \int_V \text{div}(\vec{E}\times\vec{H})dV$$

Or

$$\int_V \vec{E}.\vec{J}dV = -\int_V \left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right]dV - \int_S (\vec{E}\times\vec{H}).ds$$

As
$$\vec{B} = \mu\vec{H}, \quad \vec{D} = \varepsilon\vec{E}$$

And
$$\int_V \text{div}(\vec{E}\times\vec{H})dV = \int_S (\vec{E}\times\vec{H}).ds$$

Or

$$\int_V \left(\vec{E}.\vec{J}\right)dV = -\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right]dV - \int_S \left(\vec{E}\times\vec{H}\right)ds$$

Or

$$\int_S \left(\vec{E}\times\vec{H}\right)ds = -\int_V \frac{\partial U_{em}}{\partial t}dV - \int_V \left(\vec{E}.\vec{J}\right)dV$$

$$\int_S \vec{P}.ds = -\int_V \frac{\partial U_{em}}{\partial t}dV - \int_V \left(\vec{E}.\vec{J}\right)dV \quad (\text{as } \vec{P} = \vec{E}\times\vec{H}) \quad ...(9)$$

i.e., Total power leaving the volume = rate of decrease of stored e.m. energy – Ohmic power dissipated due to charge motion

This equation (9) represents the **Poynting theorem** according to which the net power flowing out of a given volume is equal to the rate of decrease of stored electromagnetic energy in that volume minus the conduction losses.

In equation (9) $\int_S \vec{P}.ds$ represents the amount of electromagnetic energy crossing the closed surface per second or the rate of flow of outward energy through the surface S enclosing volume V i.e., it is Poynting vector.

The term $\int_V \frac{\partial U_{em}}{\partial t} dV$ or $\frac{\partial}{\partial t}\int_V \left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] dV$ the term $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum denotes the total energy stored in electromagnetic field. So total terms gives the rate of decrease of energy stored

This is also known as work-energy theorem. This is also called as the energy conservation law in electromagnetism.

In volume V due to electric and magnetic fields, $\int_V (\vec{E}\vec{J}) dV$ gives the rate of energy transferred into the electromagnetic field.

Complex Poynting vector $\Rightarrow$ $\vec{P} = \vec{E} \times \vec{H}$

Whereas the direction of poynting vector is

$\vec{P} = a_E \times a_H$

The time-average power can be obtained from the complex Poynting vector using Pav = 1/2 Re{E×H }.

Significance of the Poynting vector in a Static Electromagnetic Field

In an electromagnetic field the flow of energy is given by the Poynting vector. For an electromagnetic wave, this vector is in the direction of propagation and accounts for radiation pressure. However, in a static electromagnetic field the Poynting vector can of course be non-zero.

Ques 5) Determine the power flow in a coaxial cable.

Ans: Power Flow in a Co-Axial Cable

Consider a co-axial cable which has a DC voltage 'V' between the conductors and a steady current I flowing in the inner and outer conductors.

The radius of inner and outer conductor are 'a' and 'b' respectively,

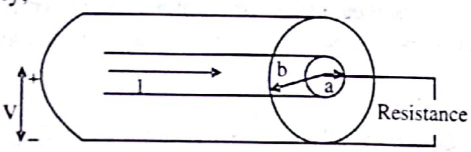


Figure 5.2

By Ampere's law,

$$\int H \cdot dL = I$$

$\int dL$ = Circumference of circular path between a and b = $2\pi r$

$$H \cdot (2\pi r) = I$$

$$H = \frac{1}{2\pi r} \quad a < r < b$$

E due to an infinitely long conductor

$$E = \frac{\lambda}{2\pi\varepsilon r} \quad \text{..... (1)}$$

Where, λ is the charge density.

The potential difference between the conductors is

$$V = \frac{\lambda}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right) \quad \text{..... (2)}$$

E in terms of V from equation (1) and (2) is

$$E = \frac{V}{\ln\left(\frac{b}{a}\right) r}$$

Power density $P = E \times H$

Since E and H are always perpendicular to each other

$$P = E \cdot H$$

$$P = \frac{V}{\ln\left(\frac{b}{a}\right) r} \cdot \frac{I}{2\pi r}$$

The total power will be given by the integration of power density P over any cross-section surface.

Let the elemental surface already be $2\pi r dr$

$$\text{Total power } W = \int \frac{V}{\ln\left(\frac{b}{a}\right) r} \cdot \frac{I}{2\pi r} (2\pi r) dr$$

$$W = \frac{V}{\ln\left(\frac{b}{a}\right)} I \int_a^b \frac{1}{r} dr$$

$$W = \frac{V}{\ln\left(\frac{b}{a}\right)} \cdot I\left(\ln\frac{b}{a}\right)$$

$$W = VI$$

i.e., the power flow along the cable is the product of V and I.

Ques 6) Discuss the average, instantaneous and complex poynting vector.

Ans: Instantaneous Poynting Vector

In electromagnetic field theory, the relations between the Poynting vector and the field strength are very much similar to those relations between power and voltage and current in A.C. circuits.

In general, the Poynting vector is given by,

$$\vec{P} = \vec{E} \times \vec{H} \quad ...(1)$$

Equation (1) represents the instantaneous power flow per unit area. Hence it is also called **instantaneous** Poynting vector.

The complex Poynting Vector is given by,

$$\vec{P} = \frac{1}{2}\vec{E} \times \vec{H} \quad .....(2)$$

From equation (2) it is clear that the product of $\vec{E}$ and $\vec{H}$ is a vector product. The mutually perpendicular components of $\vec{E}$ and $\vec{H}$, contribute to the power flow. This power flaw is directed along the normal to the plane containing $\vec{E}$ and $\vec{H}$. Thus the complex flow of power per unit area normal to the x-y plane is given by,

$$P_z = \frac{1}{2}(E_x H_y^* - E_y H_x^*) \quad .....(3)$$

Using the complex Poynting vector, the average and reactive parts of the power flow per unit area can be obtained.

Average Poynting Vector

The average part of the power flow per unit area is given by,

$$\vec{P}_{avg} = \frac{1}{2} Re[\vec{E} \times \vec{H}^*] \quad .....(4)$$

Similarly the reactive part of the power flow per unit area is given by,

$$\vec{P}_{react} = \frac{1}{2} I_m[\vec{E} \times \vec{H}^*] \quad .....(5)$$

Complex Poynting Vector

In term of the complex Poynting vector, the total instantaneous Poynting vector can be written as,

$$\vec{P}_{inst} = \vec{P} = \vec{P}_{avg} + \vec{P}_{react}$$

$$\therefore \quad \vec{P} = \frac{1}{2} Re[\vec{E} \times \vec{H}^*] + \frac{1}{2} I_m[\vec{E} \times \vec{H}^*] \quad .....(6)$$

Ques 7) For a wave travelling in air, the electric field is given by $\overline{E} = 6 \cos(\omega t - \beta z)\ \vec{a}_x$ at frequency 10MHz. Calculate:

1) **β,**
2) **$\overline{H}$, and**
3) **Average Poynting vector.**

Ans:

1) For air as a medium, the velocity of propagation is

$$v = c = 3 \times 10^8 \text{ m/s}$$

Then the wavelength is given by,

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^6} = 30\text{m}$$

Hence phase constant β is given by,

$$\beta = \frac{2\pi}{\lambda} = \frac{2\times\pi}{30} = 0.2094 \text{ rad/m}$$

2) For air, the intrinsic impedance is given by,

$$\eta = \eta_o = 120\pi = 377\Omega$$

The electric field $\vec{E}$ and the magnetic field $\vec{H}$ are in phase quadrature. As $\vec{E}$ is in x-direction, $\vec{H}$ must be in y-direction so that the wave travels in z-direction.

$$\therefore \ H = \frac{\vec{E}}{\eta_o} = \frac{6}{377}\cos(\omega t - \beta z) - \vec{a}_y \text{ A/m}$$

3) The average Poynting vector is given by,

$$\vec{P}_{avg} = \frac{1}{2} Re[\vec{E} \times \vec{H}^*]$$

Let us represent $\overline{E}$ in phasor form as,

$$\vec{E} = 6\, e^{j(\omega t - \beta z)}\, \vec{a}_x$$

Similarly $\vec{H}$ in phasor form can be represented as,

$$\vec{H} = \frac{6}{377} e^{j(\omega t - \beta z)}\, \vec{a}_y$$

The complex conjugate of $\overline{H}$ can be written as,

$$\vec{H}^* = \frac{6}{377} e^{-j(\omega t - \beta z)}\, \vec{a}_y$$

Hence average Poynting vector is given by.

$$\vec{P}_{avg} = \frac{1}{2}\left\{[6e^{j(\omega t - \beta z)}\,\vec{a}_x] \times \left[\frac{6}{377} e^{-j(\omega t - \beta z)}\,\vec{a}_y\right]\right\}$$

$$= \frac{1}{2}\left(\frac{36}{377}\right)(\vec{a}_x \times \vec{a}_y)$$

$$= 0.0477\ \vec{a}_z \text{ watt/m}^2$$

Ques 8) In a Non-magnetic Medium $E = 4\sin(2\pi \times 10^7 t - 0.8x)a_z$ V/m. Find

1) **ε_r, η.**
2) **The time-average power carried by the wave.**
3) **The total power crossing 100cm^2 of plan $2x + y = 5$.**

Ans:

1) Since $\alpha = 0$ and $\beta \neq \omega/c$, the medium is not free space but a lossless medium.

$\beta = 0.8$, $\omega = 2\pi \times 10^7$, $\mu = \mu_0$ (nonmagnetic),
$\varepsilon = \varepsilon_0 \varepsilon_r$

Hence, $\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$

$$\text{Or } \sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8(3\times10^8)}{2\pi\times10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_0}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi\times\frac{\pi}{12} = 10\pi^2$$

$$= 98.7\ \Omega$$

2) $$P = E\times H = \frac{E_0^2}{\eta}\sin^2(\omega t - \beta x)a_x$$

$$P_{ave} = \frac{1}{T}\int_0^T P\,dt = \frac{E_0^2}{2\eta}a_x = \frac{16}{2\times10\pi^2}a_x$$

$$= 81\ a_x\ \text{mW/m}^2$$

3) On place $2x + y = 5$

$$a_n = \frac{2a_x + a_y}{\sqrt{5}}$$

Hence the total power is

$$P_{ave} = \int P_{ave}\cdot dS = P_{ave}\cdot S a_n$$

$$= (81\times10^{-3}a_x)\cdot(100\times10^{-4})\left[\frac{2a_x + a_y}{\sqrt{5}}\right]$$

$$= \frac{162\times10^{-5}}{\sqrt{5}} = 724.5\mu W$$

Ques 9) A plane wave is travelling in a medium for which $\sigma = 0$, $\mu_r = 2$ and $\varepsilon_r = 4$. If the average Poynting vector is $5W/m^2$. Find:
1) **Phase velocity,**
2) **Intrinsic impedance,**
3) **R.M.S. value of E, and**
4) **R.M.S. value of H.**

Ans: For the given medium $\sigma = 0$, assuming medium to be a lossless medium,

1) The phase velocity of the wave is given by

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{(\mu_0\mu_r)(\varepsilon_0\varepsilon_r)}}$$

Substituting the values of μ_0, μ_r, ε_0 and ε_r,

$$v = \frac{1}{\sqrt{(4\times\pi\times10^{-7}\times2)(8.854\times10^{-12}\times4)}}$$

$$= 1.0599\times10^8\ m/s$$

2) The intrinsic impedance at the medium is given by,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad ...\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$$

$$\therefore \quad \eta = (377)\sqrt{\frac{2}{4}}$$

$$\therefore \quad \eta = 266.58\Omega$$

3) The magnitude of the average Poynting vector is given by,

$$\left|\vec{P}_{avg}\right| = \frac{1}{2}\frac{E_m^2}{\eta}$$

$$\therefore \quad 5 = \frac{1}{2}\frac{E_m^2}{266.58}$$

$$\therefore \quad E_m^2 = 2665.8$$

$$\therefore \quad E_m = 51.6313\ \text{V/m}$$

Thus the r.m.s. value of the electric field is given by,

$$E_{r.m.s.} = \frac{E_m}{\sqrt{2}} = \frac{51.6313}{\sqrt{2}} = 36.5\ \text{V/m}$$

4) The r.m.s. value of the magnetic field is given by,

$$H_{r.m.s.} = \frac{E_{r.m.s.}}{\eta} = \frac{36.5}{266.58} = 136.91\,\text{m A/m}$$

ELECTROMAGNETIC WAVES

Ques 10) Determine the electromagnetic wave equation from maxwell's equation.

Ans: Electromagnetic Wave Equation from Maxwell's Equation

Maxwell was the first to note that Ampere's law does not satisfy conservation of charge (his corrected form is given in Maxwell's equation). This can be shown using the equation of conservation of electric charge,

$$\nabla\cdot J + \frac{\partial\rho}{\partial t} = 0$$

Now consider Faraday's Law in differential form,

$$\nabla\times E + \frac{\partial B}{\partial t}$$

Taking the curl on both sides of above equation,

$$\nabla\times(\nabla\times E) = \nabla\times\left(-\frac{\partial B}{\partial t}\right)$$

The right-hand side may be simplified by noting that,

$$\nabla\times\left(\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times B)$$

Recalling Ampere's Law,

$$-\frac{\partial}{\partial t}(\nabla\times B) = -\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

Therefore

$$\nabla\times(\nabla\times E)=-\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

The left hand side may be simplified by the following vector identity,

$$\nabla\times(\nabla\times E)=-\nabla^2 E$$

Hence,

$$\nabla^2 E=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} \quad ...(1)$$

Applying the same analysis to Ampere's Law then substituting in Faraday's Law leads to the result

$$\nabla^2 B=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} \quad ...(2)$$

Thus the equation (1) and (2) are the electromagnetic wave equation.

Ques 11) What is uniform plane wave? Explain. Also mention the properties.

Or

Mention the properties of uniform plane wave.

Ans: Uniform plane wave

Uniform plane wave is a kind of TEM wave which come across very often in wave propagation. The wave at the receiving antenna, separated by a large distance from the transmitting antenna is considered a uniform plane wave.

It is a wave whose equiphase surfaces are planes. A plane wave has no electric or magnetic-field components along its direction of propagation.

A wave in which the electric and magnetic field intensities are directed in fixed directions in space and are constant in magnitude and phase on planes perpendicular to the direction of propagation. It is characterized by electric and magnetic fields that have uniform properties at all points across an infinite plane.

For a field to be constant in amplitude and phase on infinite planes, the source must also be infinite in extent. In this sense a plane wave cannot be generated in practice. However in many practical situations, like signal form the far away T.V. transmitter or signal of a satellite at its earth station wave can be approximated to a plane wave.

Properties of Uniform Plane Wave

Uniform plane wave exhibits certain important properties which are as follows:

1) In UPWs, there cannot be any component of electric or magnetic field along the direction of propagation of the wave. It implies that both the electric vector and the magnetic vector must be entirely normal to the direction of propagation.

2) The expressions for fields in uniform plane wave travelling along z direction in time domain for non-conductive medium are the functions of $(z - v_0t)$. The x component of electric field in UPW in non-conductive medium, travelling along z direction in time domain appear as: .....(1)

$$E_x = f_1(z - vt) + f_2(z + vt)$$

3) The expressions for fields in uniform plane wave in phasor form are for the electric vector of a wave travelling in z^+ direction, it is $E(z) = E_0e^{-j\beta z}$ and for a wave travelling in z^- direction, it is $E(z) = E_0e^{j\beta z}$ for non-conductive medium. In case of conductive medium, for a wave travelling in z^+ direction, it is $E(z) = E_0e^{-\gamma z}$ and for a wave travelling in z^- direction, it is $E(z) = E_0e^{\gamma z}$.

4) In case of wave travelling in arbitrary direction, it requires the use of direction cosines to write expressions for fields in uniform plane wave in phasor domain as:

$$E(r)=E_0e^{-j\beta\hat{n}\cdot r}=E_0e^{-j\beta(x\cos A+y\cos B+z\cos C)} \quad .... (2)$$

If the medium has conductivity, then

$$E(r)=E_0e^{-\gamma\hat{n}\cdot r}=E_0e^{-\gamma(x\cos A+y\cos B+z\cos C)} \quad .....(3)$$

Here, A, B and C are the angles, the ray or wave normal makes with x, y and z axes, respectively. The cosines of the angles, i.e., Cos A, Cos B and Cos C are called direction cosines of the wave.

5) The ratio of the electric vector E to the magnetic vector H for UPW in free space is given by

$$\frac{E}{H}=\sqrt{\frac{\mu}{\varepsilon}} \quad .....(4)$$

If the medium happens to be with non-zero conductivity, then it can be shown that its intrinsic impedance is:

$$\frac{E}{H}=\sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}} \quad .....(5)$$

6) The electric and magnetic fields remain in time phase, i.e., reach their maximum values, minimum values or mean values at the same time, i.e., simultaneously at any point of UPW.

7) The relative orientation of electric vector E and magnetic vector H is normal to each other, and the direction of the wave propagation is same as the direction of the vector E × H. The electric vector E, magnetic vector H and the direction of propagation form a right-hand vector system.

8) A UPW is associated with the power flow. The amount of power flow per unit area is:

$$P=\frac{E^2}{2\eta}W/m^2 \quad .....(6)$$

9) A UPW is also associated with the energy storage and the average energy density in the wave is:

$$W_{ave} = \frac{\varepsilon E^2}{2} = \frac{\mu H^2}{2} J/m^3 \quad .....(7)$$

Ques 12) A uniform plane wave at frequency of 300MHz travels in vacuum along + y direction. The electric field of the wave at some instant is given as $E = 3\hat{x} + 5\hat{z}$. Find the phase constant of the wave and also the vector magnetic field.

Ans: The phase constant

$$\beta = \omega\sqrt{\mu \epsilon} = \frac{2\pi \times 300 \times 10^6}{3 \times 10^8} = 2\pi \text{rad/m}$$

The wave is travelling along +y direction. Therefore, the H vector should lie in xz– plane. Let the vector magnetic field be given by $H = A\hat{x} + B\hat{z}$. Since for a uniform plane wave E and H are perpendicular to each other,

$$E \cdot H = 0$$

$$\Rightarrow 3A + 5B = 0 \quad .....(1)$$

Since, $\frac{|E|}{|H|} = \eta = 120\pi\ \Omega$

$$\Rightarrow |H| = \sqrt{A^2 + B^2} = \frac{|E|}{\eta} = \frac{\sqrt{9+25}}{120\pi} \text{ A/m} \quad .....(2)$$

$$= 15.46 \times 10^{-3}$$

Solving equation (1) and (2),

$$A = \pm\frac{5}{\sqrt{120\pi}} \text{ and } B = \mp\frac{3}{\sqrt{120\pi}}$$

The vector magnetic field is

$$H = \frac{5\hat{x} - 3\hat{z}}{\sqrt{120\pi}} \text{ A/m}$$

Ques 13) Discuss the reflection of plane electromagnetic waves at boundaries for normal incidence.

Or

Explain the reflection of plane wave for the normal incidence. Discuss about reflection and transmission coefficient for E and H.

Or

Explain the terms transmission coefficient and reflection coefficient.

Or

Define reflection coefficient of a plane wave in a normal incidence.

Ans: Reflection of a Plane EM Waves in a Normal Incidence

When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted. The proportion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ε, μ, σ) of the two media involved.

Suppose that a plane wave propagating along the +z-direction is incident normally on the boundary z = 0 between medium 1 (z < 0) characterised by σ_1, ε_1, μ_1 and medium 2(z > 0) characterised by σ_2, ε_2, μ_2, as shown in **figure 5.3**. In the figure, subscripts i, r, and t denote incident, reflected, and transmitted waves, respectively.

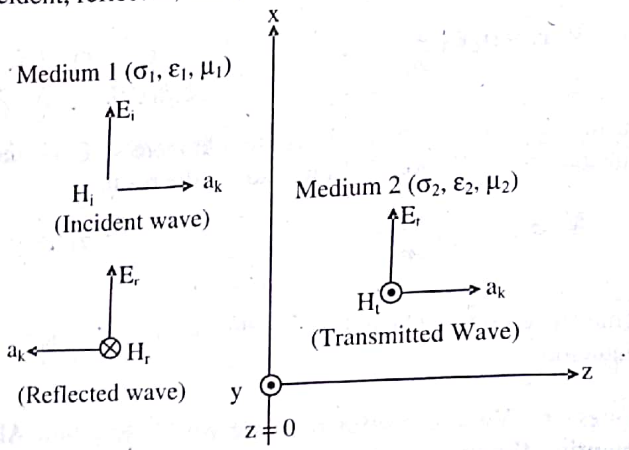


Figure 5.3: A Plane Wave Incident Normally on an Interface between Two Different Media

The incident, reflected, and transmitted waves shown in **figure 5.3** are obtained as follows:

Incident Wave

(E_i, H_i) is traveling along $+a_z$ in medium 1. If we suppress the time factor $e^{j\omega t}$ and assume that

$$E_{is}(z) = E_{i0}e^{-\gamma_1 z}a_x \quad .....(1)$$

Then

$$H_{is}(z) = H_{i0}e^{-\gamma_1 z}a_y = \frac{E_{i0}}{\eta_1}e^{-\gamma_1 z}a_y \quad .....(2)$$

Reflected Wave

(E_r, H_r) is traveling along $-a_z$ in medium 1. If

$$E_{rs}(z) = E_{r0}e^{\gamma_1 z}a_x \quad .....(3)$$

Then,

$$H_{rs}(z) = H_{r0}e^{\gamma_1 z}(-a_y) = -\frac{E_{r0}}{\eta_1}e^{\gamma_1 z}a_y \quad .....(4)$$

Where E_{rs} has been assumed to be along a_x; we will consistently assume that for normal incident E_i, E_r, and E_t have the same polarization.

Transmitted Wave

(E_t, H_t) is traveling along $+a_z$ in medium 2. If,

$$E_{ts}(z) = E_{t0}e^{-\gamma_2 z}a_x \quad .....(5)$$

Then, $H_{ts}(z) = H_{t0}e^{-\gamma_2 z}a_y = \frac{E_{t0}}{\eta_2}e^{-\gamma_2 z}a_y \quad .....(6)$

In equations (1) to (6), E_{i0}, E_{r0}, and E_{t0} are, respectively, the magnitudes of the incident, reflected, and transmitted electric fields at z = 0.

Notice from **figure 5.3** that the total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, i.e.,

$E_1 = E_i + E_r, \qquad H_1 = H_i + H_r$

$E_2 = E_t, \qquad H_2 = H_t$

At the interface z = 0, the boundary conditions require that the tangential components of E and H fields must be continuous. Since the waves are transverse, E and H fields are entirely tangential to the interface.

Hence at z = 0, $E_{1tan} = E_{2tan}$ and $H_{1tan} = H_{2tan}$ imply that

$$E_i(0) + E_r(0) = E_t(0) \Rightarrow \quad E_{i0} + E_{r0} = E_{t0} \quad .....(7)$$

$$H_i(0) + H_r(0) = H_t(0) \Rightarrow \quad \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2} \quad .....(8)$$

From equations (7) and (8), we obtain

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \quad ..... (9)$$

$$\text{And, } E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0} \quad ..... (10)$$

We now define the reflection coefficient Γ and the transmission coefficient τ from equations (9) and (10) as

Reflection coefficient, $\Gamma = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ .....(11 a)

Or, $E_{r0} = \Gamma E_{i0}$ .....(11 b)

Transmission coefficient, $\tau = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ ......(12 a)

Or, $E_{t0} = \tau E_{i0}$ ......(12 b)

Note:
1) $1 + \Gamma = \tau$
2) Both Γ and τ are dimensionless and may be complex. $0 \le |\Gamma| \le 1$

Ques 14) A uniform plane wave travels in +x direction. The phasor electric field for the wave is $(3\hat{y} + j5\hat{z})$ V/m. Find the equation of the ellipse of polarisation. Find the maximum magnitude of the field. Also find the sense of rotation.

Ans: The wave travels in +x direction and therefore the E-vector should lie in the yz plane. The two field components in y and z directions respectively can be written as

$$E_y = 3\cos\omega t$$

$$E_z = 5\cos\left(\omega t + \frac{\pi}{2}\right)$$

Note: j in z component indicates a phase difference of $+\frac{\pi}{2}$ between E_z and E_y.

The equation of the ellipse is

$$\frac{E_y^2}{9} + \frac{2\times E_y \times E_z \cos\left(\frac{\pi}{2}\right)}{15} + \frac{E_z^2}{25} = \sin^2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{E_y^2}{9} + \frac{E_z^2}{25} = 1$$

The ellipse of polarisation is shown in **figure 3.12.**

The maximum field magnitude = 5V/m.

To find the sense of rotation, let us find the resultant field direction at say t = 0 and t = Δt (Positive).

At t = 0, $E_y = 3$ and $E_x = 5\cos\frac{\pi}{2} = 0$. So, the field vector is vertically upwards.

At t = Δt, $E_y = 3\cos\omega\Delta t = +ve$

and $\quad E_z = 5\cos(\omega\Delta t + \frac{\pi}{2}) = -ve$

The resultant vector, therefore, moves leftward. This wave is travelling in +x-direction (Inside the paper) and hence the sense of rotation is Left Handed

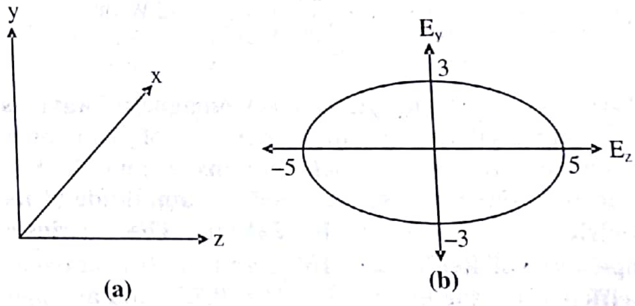


Figure 3.12: (a) Coordinate Axes (b) Ellipse of Polarisation

Ques 15) A uniform plane wave travelling along positive z direction in air strikes normally on the surface of a dielectric with μ = μ₀ and ε = 6.25ε₀. The amplitude of electric field of the incident wave is 10 V/m. Calculate the amplitudes of electric field intensities associated with the reflected and transmitted waves, assuming that the dielectric extends to infinity. Also, calculate the power per unit area carried by each wave.

Ans: Amplitude of electric field in the incident wave, $E^*_{I\times 0} = 10.0$ V/m

Intrinsic impedance of air, $\eta_1 = \eta_0 = 377\Omega$

Intrinsic impedance of the dielectric,

$$\eta_2 = \sqrt{\frac{\mu_0}{6.25\varepsilon_0}} = \frac{377}{2.5} = 150.8\Omega$$

Reflection coefficient,

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{150.8 - 377}{150.8 + 377} = -0.4286$$

Transmission coefficient = $1 + \Gamma = 1 - 0.4286$
$= 0.5714$

Amplitude of electric field in the reflected wave,

$$E^-_{1x0} = |\Gamma| E^+_{1x0} = 4.286 \text{ V/m}$$

Amplitude of electric field in the transmitted wave,

$$E^+_{2x0} = |1 + \Gamma| E^+_{1x0} = 5.714 \text{ V/m}$$

Powers per unit area carried by the waves are,

Incident wave:

$$P^+_{1.av} = \frac{(E^+_{1x0})^2}{2\eta_1} = \frac{10^2}{2 \times 377} = 0.1326 \text{ W/m}^2$$

Reflected wave: $P^-_{1.av} = \frac{(E^-_{1x0})^2}{2\eta_1} = \frac{4.286^2}{2 \times 377} = 0.0244 \text{ W/m}^2$

Transmitted wave:

$$P^+_{2.av} = \frac{(E^+_{2x0})^2}{2\eta_2} = \frac{5.714^2}{2 \times 150.8} = 0.1082 \text{ W/m}^2$$

Ques 16) A uniform plane electromagnetic wave is incident normally at the interface of two non-magnetic, perfect dielectric regions 1 and 2. The incident wave is in Region I and the amplitude of its electric field intensity is 24V/m. The intrinsic impedance of Region 2 is 188Ω and the transmission coefficient for the magnetic field is 0.75. Find average power per unit area in the incident, reflected, and transmitted waves.

Ans: Intrinsic impedance of Region 2 is $\eta_2 = 188\Omega$
Amplitude of electric field of the incident wave,

$$E^+_{10} = 24 \text{ V/m}$$

Transmission coefficient for the magnetic field,

$$= \frac{2\eta_1}{\eta_1 + \eta_2} = \frac{2\eta_1}{\eta_1 + 188} = 0.75$$

By solving, $\eta_1 = 112.8\Omega$

Reflection coefficient for the electric field,

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{188 - 112.8}{188 + 112.8} = 0.25$$

Average power per unit area in the incident wave,

$$P^+_{1,av} = \frac{E^{+2}_{10}}{2\eta_1} = \frac{24^2}{(2)(112.8)} = 2.553 \text{W/m}^2$$

Average power per unit area in the reflected wave,

$$P^-_{1,av} = \Gamma^2 P^+_{1,av} = (0.25)^2 (2.553) = 0.16 \text{ W/m}^2$$

Average power per unit area in the transmitted wave,

$$P^+_{2,av} = P^+_{1,av} - P^{-1}_{1,av} = 2.393 \text{ W/m}^2$$

Ques 17) Discuss and derive the wave equations in phasor form.

Ans: Wave Equations in Phasor Form

An electromagnetic wage in a medium can be completely defined if intrinsic impedance (η) and propagation constant (γ) of a medium is known. Thus it is necessary to derive the expressions for η and γ in terms of the properties of a medium such as permeability (μ), permittivity (ε), conductivity (σ), etc.

Consider Maxwell's equation derived from Faraday's law,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \quad .....(1)$$

Taking curl on both the sides of the equation,

$$\therefore \quad \nabla \times \nabla \times \vec{E} = -\mu \left[\nabla \times \frac{\partial \vec{H}}{\partial t} \right]$$

$$\therefore \quad \nabla \times \nabla \times \vec{E} = -\mu \left[\frac{\partial}{\partial t} (\nabla \times \vec{H}) \right] \quad .....(2)$$

Using vector identity to the left of equation (2),

$$\therefore \quad \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \left[\frac{\partial}{\partial t} (\nabla \times \vec{H}) \right] \quad .....(3)$$

But according to another Maxwell's equation,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Putting value of $\nabla \times \vec{H}$ in equation (3),

$$\therefore \quad \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \left[\frac{\partial}{\partial t} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \right] \quad .....(4)$$

Since most of the regions are source or charge free,

$$\therefore \quad \nabla \cdot \vec{E} = 0$$

$$\therefore \quad \nabla(\nabla \cdot \vec{E}) = 0$$

Putting value of $\nabla(\nabla \cdot \vec{E})$ in equation (4), assuming charge-free medium.

$$-\nabla^2 \vec{E} = -\mu \left[\frac{\partial}{\partial t} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \right]$$

Making both sides positive,

$$\nabla^2 \vec{E} = \mu\left[\frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\right] \quad .....(5)$$

Consider a general electromagnetic wave with both the fields, $\vec{E}$ and $\vec{H}$ varying with respect to time. When any field varies with respect to time, its partial derivative taken with respect to time can be replaced by jω Rewriting equation (5) in phasor form.

$$\therefore \quad \nabla^2 \vec{E} = \mu[j\omega(\vec{J} + j\omega\vec{D})]$$

$$\therefore \quad \nabla^2 \vec{E} = j\omega\mu[(\sigma\vec{E}) + j\omega(\varepsilon\vec{E})]$$

$$\therefore \quad \nabla^2 \vec{E} = [j\omega\sigma\mu\vec{E} + (j\omega)\varepsilon\mu\vec{E}]$$

$$\therefore \quad \nabla^2 \vec{E} = [j\omega\mu(\sigma + j\omega\varepsilon)]\vec{E} \quad .....(6)$$

In similar way, one can write another phasor equation as,

$$\nabla^2 \vec{H} = [j\omega\mu(\sigma + j\omega\varepsilon)]\vec{H} \quad .....(7)$$

The terms inside the bracket in equations (6) and (7) are exactly similar and represent the properties of the medium in which wave is propagating. The total bracket is the square of a propagation constant γ, hence re- equations (6) and (7) can be rewritten as,

$$\nabla^2 \vec{E} = \gamma^2 \vec{E} \text{ and}$$

$$\nabla^2 \vec{H} = \gamma^2 \vec{H}$$

So the propagation constant γ can be expressed in terms of properties of the medium as,

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \quad .....(8)$$

The real and imaginary parts of γ are attenuation constant (α) and phase constant (β) and both can be expressed in terms of the properties of the medium,

$$\therefore \quad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)} \quad .....(9)$$

$$\text{and,} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)} \quad .....(10)$$

Ques 18) Calculate the attenuation constant and phase constant for a uniform plane wave with frequency of 10GHz in polythelene for which $\mu = \mu_0$, $\varepsilon_r = 2.3$ and $\sigma = 256 \times 10^{-4}$ Ʊ/m.

Ans: The propagation constant in loss dielectric is given by

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\therefore \quad \gamma = \sqrt{j(2\pi f)(\mu_0\mu_r)[\sigma + j(2\pi f)(\varepsilon_0\varepsilon_r)]}$$

$$\therefore \gamma = \sqrt{\begin{aligned}&j(2\times\pi\times10\times10^9\times4\times\pi\times10^{-7})[2.56\times10^{-4}\\&+ j(2\times\pi\times10\times10^9(8.854\times10^{-12}\times2.3)]\end{aligned}}$$

$$\therefore \quad \gamma = \sqrt{j(78.9568\times10^3)[2.56\times10^{-4} + j1.2795]}$$

$$\therefore \quad \gamma = \sqrt{[78.9568\times10^3\angle 90°][1.2795\angle 89.98°]}$$

$$\therefore \quad \gamma = \sqrt{101.025\times10^3\angle 179.98°}$$

$$\therefore \quad \gamma = 317.84\angle 89.99° = 0.0554 + j317.84$$

Thus, $\alpha = 0.0554$ Np/m
$\beta = 317.84$ rad/m

Module 6

Wave Propagation and Transmission Lines

WAVE PROPAGATION

Ques 1) Discuss the wave propagation in loss dielectrics. Also determine the E and H for the loss dielectrics.

Or

Explain the wave propagation of plane electromagnetic wave in lossy dielectric (Conducting media).

Ans: Wave Propagation of Plane Electromagnetic Wave in Lossy Dielectric

Wave propagation in lossy dielectrics is a general case from which wave propagation in other types of media can be derived as special cases. A lossy dielectric is a medium in which an EM wave loses power as it propagates due to poor conduction.

In other words a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$

Wave propagation in general medium i.e., lossy dielectric medium (σ, ε, μ)

Since,

$$\nabla \times H_s = \sigma E_s + j\omega\varepsilon E_s \qquad .....(1)$$

$$\nabla \times E_s = -j\omega\mu H_s \qquad .....(2)$$

Taking curl of equation (2)

$$\nabla \times (\nabla \times E_s) = -j\omega\mu(\nabla \times H_s)$$

Put the value of $\nabla \times H_s$ from equation(1)

$$\nabla (\nabla \cdot E_s) - \nabla^2 E_s = -j\omega\mu\ (\sigma E_s + j\omega\varepsilon E_s)$$

$$0 - \nabla^2 E_s = -j\omega\mu\ (\sigma E_s + j\omega\varepsilon E_s)$$

$$\Rightarrow \quad \nabla^2 E_s = j\omega\mu\ (\sigma E_s + j\omega\varepsilon E_s)$$

$$\text{Or} \quad \nabla^2 E_s = j\omega\mu\ (\sigma + j\omega\varepsilon E_s)$$

$$\text{Or} \quad \nabla^2 E_s - j\omega\mu\ (\sigma + j\omega)\ \varepsilon E_s = 0$$

$$\nabla^2 E_s - \gamma^2 E_s = 0 \qquad .....(3)$$

Where, γ is propagation constant and given by,

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \qquad .....(4)$$

And, α = Attenuation constant, β = Phase constant

Assumptions

1) Electric wave has amplitude in only x-direction.
2) Wave is moving in z-direction.

Since, $E = E_x\, a_x + E_y\, a_y + E_z\, a_z$

From assumptions,

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} - \gamma^2 E_{xs} = 0$$

$$\Rightarrow \frac{d^2 E_{xs}}{dz^2} - \gamma^2 E_{xs} = 0$$

$$\Rightarrow (D^2 - \gamma^2)E_{xs} = 0$$

On solving

$$E_{xs} = E_0 e^{-\gamma z} + E_0 e^{\gamma z} \qquad .....(5)$$

Where, $E_0 e^{-\gamma z}$ = Wave travelling in the +ve direction.
$E_0 e^{\gamma z}$ = Wave returning i.e., in –ve direction.

Since, wave is moving towards infinity so,

$$E_{xs} = E_{x0}\, e^{-\gamma z} a_x \qquad .....(6)$$

Since $\nabla \times E_s = -j\omega\mu H_s$

$$\begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{x_s} & 0 & 0 \end{bmatrix} = -j\omega\mu H_s$$

$$a_y\left(\frac{\partial}{\partial z} E_{x_s} - 0\right) + 0 = -j\omega\mu H_s$$

$$j\omega\mu H_s = -\frac{\partial}{\partial_z} Ex_s\, a_y$$

$$j\omega\mu H_s = -\frac{\partial}{\partial_z} E_{x0} e^{-\gamma z}\, a_y$$

$$\Rightarrow \quad j\omega\mu H_s = \gamma E_{x_0} e^{-\gamma z} a_y$$

$$H_{ys} = \frac{\gamma}{j\omega\mu} E_{x_0} e^{-\gamma z} a_y$$

$$H_{ys} = \sqrt{\frac{(\sigma + j\omega\varepsilon)}{j\omega\mu}}\, E_{x_0} e^{-\gamma z} a_y$$

$$H_{ys} = \frac{E_{x_0} e^{-\gamma z} a_y}{\eta} \qquad .....(7)$$

Since, direction of propagation is given by, $a_E \times a_H = a_K$

i.e., $a_x \times a_y = a_z$ .....(8)

As we know equation of electric field in Phasor domain,

$$E_{x_s} = E_{x_0} e^{-\gamma z} a_x$$

$$= E_{x_0} e^{-(\alpha + j\beta)z} a_x$$

Or, $$E_{x_s} = E_{x_0} e^{-\alpha z} e^{-j\beta z} a_x$$

This equation can be written in time domain as,

$$E_x(z,t) = E_{x_0} e^{-\alpha z} \cos(\omega t - \beta z) a_x \text{ V/m} \quad .....(9)$$

Where, βz = Phase angle in (rad)

$$\text{And } \beta = \frac{\text{Phase angle}}{z} = \frac{\text{rad}}{m}$$

Drawing the amplitude of equation (9) w.r.t. (z) gives the following **figure 6.1**:

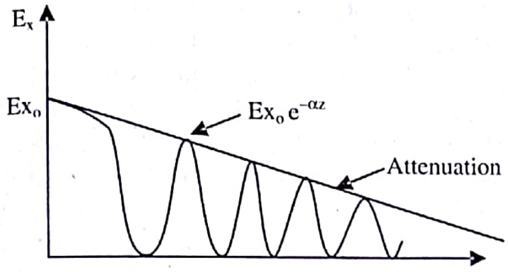


Figure 6.1

$$\Rightarrow H_y(z,t) = \frac{E_{x_0} e^{-\alpha z}}{\eta} \cos(\omega t - \beta z - \theta) ay \text{ A/m} \quad .....(10)$$

Ques 2) Explain the wave propagation of plane electromagnetic wave in lossless dielectric.

Ans: Plane Waves in Lossless Medium

In a lossless medium, ε and μ are real numbers, so k is real.

Let us consider E_x component,

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad .....(1)$$

Let us consider a plane wave which has only E_x component and propagating along z. Since the plane wave will have no variation along the plane perpendicular to z, i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation (1) reduces to,

$$\frac{\partial^2 E_x}{dz^2} + k^2 E_x = 0 \quad .....(2)$$

The solution to this equation can be written as,

$$E_x(z) = E_x^+(z) + E_x^-(z)$$

$$= E_0^+ e^{-j/z} + E_0^- e^{j/z} \quad .....(3)$$

E_x^+ and E_0^- are the amplitude constants (can be determined from boundary conditions).

In the time domain, $\varepsilon_x(z, t) = \text{Re}(E_x(z)e^{j\omega t})$

$$\varepsilon_x(z,t) = E_0^+ \cos(ax - kz) + E_0^- \cos(\omega t + kz) \quad ...(4)$$

assuming E_0^+ and E_0^- are real constants.

Here, $\varepsilon_x^+(z,t) = E_0^- \cos(\alpha t - \beta z)$ represents the forward travelling wave. The plot of $\varepsilon_x^+(z,t)$ for several values of t is shown in the **figure 6.2**.

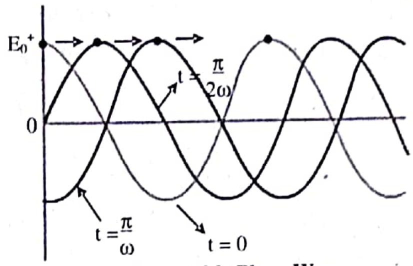


Figure 6.2: Plane Wave Traveling in the +z Direction

It can be seen from the **figure 6.2**, at successive times, the wave travels in the +z direction.

Let us fix our attention on a particular point or phase on the wave (as shown by the dot), i.e., $\omega t - kz$ = constant.

Then as t is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that,

$$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$$

or, $\omega \Delta t = k \Delta z$

$$\text{or, } \frac{\Delta z}{\Delta z} = \frac{\omega}{k}$$

When $\Delta t \to 0$,

Let us write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$ = phase velocity v_p.

$$\therefore v_p = \frac{\omega}{k} \quad .....(5)$$

If the medium in which the wave is propagating is free space, i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

$$\text{Then } v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$$

where, 'C' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points), i.e.,

$$(\omega t - kz) - [\omega t - k(z + \lambda)] = 2\pi$$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \frac{\omega}{v_p}$,

$$\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$$

or, $\lambda f = v_p$ .....(6)

Thus wavelength λ also represents the distance covered in one oscillation of the wave, Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave travelling in the -z direction.

The associated magnetic field can be found as follows: From equation (3),

$$\vec{E}_x^+(z) = E_0^+ e^{-j/z}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-j/z} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu} E_0^+ e^{-j/z}\hat{a}_y$$

$$= \frac{E_0^+}{\eta} e^{-j/z}\hat{a}_y = H_0^+ e^{-j/z}\hat{a}_y \quad .....(7)$$

Where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.

When the wave travels in free space

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$ is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^+(z,t) = \hat{a}_y \frac{E_0^+}{\eta}\cos(\omega t - \beta z) \quad .....(8)$$

Which represents the magnetic field of the wave travelling in the +z direction.

For the negative traveling wave,

$$\vec{H}^-(z,t) = -a_y \frac{E_0^+}{\eta}\cos(\omega t + \beta z) \quad .....(9)$$

For the plane waves described, both the E and H fields are perpendicular to the direction of propagation, and these waves are called TEM (Transverse Electromagnetic) waves.

Ques 3) Explain the wave propagation of plane electromagnetic wave in conducting medium.

Ans: Electromagnetic Wave in Conducting Medium

Let us consider a plane electromagnetic wave travelling in a linear dielectric medium such as air along the z direction and being incident at a conducting interface. The medium will take to be a linear medium, so that one can describe the electrodynamics using only the E and H vectors.

To investigate the propagation of the wave in the conducting medium, as the medium is linear and the propagation takes place in the infinite medium, the vectors $\vec{E}$, $\vec{H}$ and $\vec{k}$ are still mutually perpendicular. Let us take the electric field along the x direction, the magnetic field along the y- direction and the propagation to take place in the z direction. Further, take the conductivity to be finite and the conductor to obey Ohm's law, $\vec{J} = \sigma\vec{E}$.

Consider the pair of curl equations of Maxwell equation,

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \in\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} + \in\frac{\partial\vec{E}}{\partial t}$$

Let us take $\vec{E}$, $\vec{H}$ and $\vec{k}$ to be respectively in x, y and z direction. Thus the n have,

$$(\nabla\times\vec{E})_y = -\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

i.e., $\frac{\partial E_x}{\partial z} + \mu\frac{\partial H_y}{\partial t} = 0$ .....(1)

and $(\nabla\times\vec{H})_x = -\frac{\partial H_y}{\partial z} = \sigma E_x + \in\frac{\partial E_x}{\partial t}$

i.e., $\frac{\partial H_y}{\partial z} + \sigma E_x + \in\frac{\partial E_x}{\partial t} = 0$ .....(2)

Let us take the time variation to be harmonic ($\sim e^{j\omega t}$) so that the time derivative is equivalent to a multiplication by jω. The pair of Equations (1) and (2) can then be written as,

$$\frac{\partial E_x}{\partial z} + j\mu\omega H_y = 0$$

$$\frac{\partial H_y}{\partial z} + \sigma E_x + j\omega\in E_x = 0$$

Let us solve this pair of coupled equations by taking a derivative of either of the equations with respect to z and substituting the other into it,

$$\frac{\partial^2 E_x}{\partial z^2} + j\mu\omega\frac{\partial H_y}{\partial z} = \frac{\partial^2 E_x}{\partial z^2} - j\mu\omega(\sigma + j\omega\epsilon)E_x = 0$$

Define, a complex constant γ through

$\gamma^2 = j\mu\omega(\sigma + j\omega\varepsilon)$

in terms of which,

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \quad .....(3)$$

In an identical fashion,

$$\frac{\partial^2 H_y}{\partial z^2} - \gamma^2 H_y = 0 \quad .....(4)$$

Solutions of equation (3) and (4) are well known and are expressed in terms of hyperbolic functions,

$E_x = A \cos h(\gamma z) + B \sin h(\gamma z)$

$H_y = C \cos h(\gamma z) + D \sin h(\gamma z)$

Where, A, B, C and D are constants to be determined.

If the values of the electric field at z = 0 is E_0 and that of the magnetic field at z = 0 is H_0, since, $A = E_0$ and $C = H_0$.

In order to determine the constants B and D, let us return back to the original first order equations (1) and (2),

$$\frac{\partial E_x}{\partial z} + j\mu\omega H_y = 0$$

$$\frac{\partial H_y}{\partial z} + \sigma E_x + j\omega\epsilon E_x = 0$$

Substituting the solutions for E and H

$\gamma E_0 \sin h(\gamma z) + B\gamma \cos h(\gamma z) + i\omega\mu(H_0 \cos h(\gamma z) + D \sin h(\gamma z)) = 0$

This equation must remain valid for all values of z, which is possible if the coefficients of sin h and cos h terms are separately equated to zero,

$E_0\gamma + j\omega\mu D = 0$

$B\gamma + j\omega\mu H_0 = 0$

The former gives,

$$D = -\frac{\gamma}{j\omega\mu}E_0$$

$$= -\frac{\sqrt{j\mu\omega(\sigma + j\omega\epsilon)}}{j\omega\mu}$$

$$= -\frac{E_0}{\eta}$$

Where,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

Likewise,

$B = -\eta H_0$

Substituting these, our solutions for the E and H become,

$E_x = E_0 \cos h(\gamma z) - \eta H_0 \sin h(\gamma z)$

$$H_y = H_0 \cos h(\gamma z) - \frac{E_0}{\eta}\sin h(\gamma z)$$

The wave is propagating in the z direction. Let us evaluate the fields when the wave has reached $z = \ell$,

$E_x = E_0 \cos h(\gamma\ell) - \eta H_0 \sin h(\gamma\ell)$

$$H_y = H_0 \cos h(\gamma\ell) - \frac{E_0}{\eta}\sin h(\gamma\ell)$$

If ℓ is large, the one can approximate

$$\text{Sin h}(\gamma\ell) \approx \cos h(\gamma\ell) = \frac{e^{\gamma\ell}}{2}$$

Thus we have,

$$E_x = (E_0 - \eta H_0)\frac{e^{\gamma\ell}}{2}$$

$$H_y = (H_0 - \frac{E_0}{\eta})\frac{e^{\gamma\ell}}{2}$$

The ratio of the magnitudes of the electric field to magnetic field is defined as the "characteristic impedance" of the wave

$$\left|\frac{E_x}{H_y}\right| = \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

Ques 4) Write the intrinsic impedance and propagation constant in all medium.

Or

Write the equations for following medium:

1) Plane Waves in Lossless Dielectrics

2) Plane Waves in Free Space/ Perfect Dielectric

3) Plane Waves in Good Conductors

Or

Find the value of α and β for good conductors. Show that angle of charactristics impedence is always 45° for good conductors.

Ans: Plane Waves in Lossless Dielectrics

In a lossless dielectric, $\sigma << \omega\varepsilon$. It is a special case

$\sigma \approx 0, \varepsilon = \varepsilon_0\varepsilon_r, \quad \mu = \mu_0\mu_r$ .....(1)

Hence,

$\alpha = 0, \beta = \omega\sqrt{\mu\varepsilon}$ .... (2 a)

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \lambda = \frac{2\pi}{\beta} \quad .....(2\ b)$$

Also $\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0°$ .....(3)

And thus E and H are in time phase with each other and given by,

$E = E_0 \cos(\omega t - \beta z)\, a_x$ and

$H = H_0 \cos(\omega t - \beta z) a_y$

Plane Waves in Free Space/ Perfect Dielectric

In this case,

$\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$ .....(4)

This may also be regarded as a special case. Thus we simply replace ε by ε_0 and μ by μ_0 in equation (2 a and b) we obtain,

$$\sigma = 0, \quad \beta = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} \quad .....(5a)$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \quad \lambda = \frac{2\pi}{\beta} \quad .....(5b)$$

Where, $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

$$\text{Since, } |\eta| = \sqrt{\frac{\mu/\varepsilon}{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}}} = \sqrt{\frac{\mu/\varepsilon}{\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{1/4}}} \quad .....(6)$$

By substituting the constitutive parameters in equation (4) into equation (6), $\sigma = 0$ θ_η, = 0 and $\eta = \eta_0$, where η_0, is called the intrinsic impedance of free space and is given by

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega \quad ..... (7)$$

$E = E_0 \cos(\omega t - \beta z)\, a_x$ .....(8a)

Then,

$$H = H_0 \cos(\omega t - \beta z) a_y = \frac{E_0}{\eta_0} \cos(\omega t - \beta z) a_y \quad ....(8b)$$

Plane Waves in Good Conductors

A perfect, or good conductor, is one in which $\sigma >> \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$; i.e.,

$\sigma \simeq \infty, \quad \varepsilon = \varepsilon_0, \quad \mu = \mu_0\mu_r$ .....(9)

$$\text{Hence, } \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} \quad ..... (10\ a)$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \ \lambda = \frac{2\pi}{\beta} \quad ..... (10\ b)$$

Also, $\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45°$ ..... (11)

And thus E leads H by 45°. If

$E = E_0 e^{-\gamma z} \cos(\omega t - \beta z)\, a_x$ .....(12 a)

Then ,

$$H = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\gamma z} \cos(\omega t - \beta z - 45°) a_y \quad ..... (12\ b)$$

Ques 5) Find the expression for α,β and γ for loss less or perfect dielectric medium.

Ans: Propagation constant, Phase Constant and Attenuation Constant for loss less or perfect dielectric medium

The propagation constant is given by,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} m^{-1} \quad ...(1)$$

For the perfect dielectric, substituting $\sigma = 0$, $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$ in above expression we get,

$$\gamma = \sqrt{j\omega\mu(0 + j\omega\varepsilon)} \quad ...(2)$$

$$\therefore \gamma = \pm j\omega\sqrt{\mu\varepsilon}\, m^{-1} \quad .....(3)$$

Also $\gamma = \alpha + j\beta$

Hence the attenuation constant for the perfect dielectric is given by,

$\alpha = 0$ .....(4)

The phase constant for the perfect dielectric is given by,

$\beta = \omega\sqrt{\mu\varepsilon}$ rad/m .....(5)

The intrinsic impedance is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Putting $\sigma = 0$ for perfect dielectric, we get

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$\therefore \eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}}\Omega \quad .....(6)$$

Ques 6) A uniform plane wave propagating in a good conductor if the magnetic field intensity is given by, $H = 0.1\ e^{-15z}\cos(2\pi \times 10^8 t - 15z)\ i$ A/m. Determine the conductivity and corresponding component of E field. Also calculate the average power loss in a block of unit area and thickness t.

Ans: Comparing the given equation with the general equation for, $\vec{H}$, get the following terms:

$\alpha = 15\ \beta = 15\ \omega = 2\pi \times 10^8$ rad/s

$$\because \quad \alpha = \beta = \sqrt{\pi\mu f\sigma}$$

$$\sigma = \frac{\alpha^2}{\pi\mu f} = \frac{(15)^2}{\pi \times 4\pi \times 10^{-7} \times 10^8} = 0.57\,s/m$$

$$\delta = \frac{1}{\alpha} = \frac{1}{15} = 0.067\text{m}$$

Now for $\vec{E}$

Assume $\varepsilon = \varepsilon_o$

$$\because \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\mu\epsilon}} = |\eta|e^{i\theta\eta}$$

$$\because \quad |\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}};\ \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}$$

$$\therefore \quad \tan 2\theta_\eta = \frac{0.57}{2\pi \times 10^8 \times 8.85 \times 10^{-12}} = 1.025$$

$$\theta_\eta = \frac{1}{2}\tan^{-1}(1.025) = 22.85°$$

$$|\eta| = \frac{\sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}}}{\sqrt[4]{1+(1.025)^2}} = \frac{377}{1.20} = 315\Omega$$

Hence, $\vec{E}(z,t) = \eta \vec{H} \cdot \hat{a}_E$

$\because\ \hat{a}_K = \hat{a}_y;\quad \hat{a}_H = \hat{a}_x$

$\hat{a}_E \times \hat{a}_H = \hat{a}_k;\quad \hat{a}_E \times \hat{a}_x = (-\hat{a}_y);\ \hat{a}_E = (-a_y)$

$$\vec{E}(z,t) = 315 \times 0.1 \times e^{-15z}\cos(2\pi \times 10^8 t - 15z + \theta_\eta)(-\hat{a}_y)\,V/m$$

$$\vec{E}(z,t) = -31.5e^{-15z}\cos(2\pi \times 10^8 t - 15z + 22.85°)\hat{a}_y\,V/m$$

$$P_{ave} = \frac{1}{2}[E \times H^*] = \frac{1}{2} \times 0.1 \times 31.5e^{-30z}\cos 22.85°$$

$= 1.45\ e^{-30z}$ W/m^2

At z = 0, $P_{ave}(0) = 1.45$; At z = δ

$P_{ave}(\delta) = 1.45\ e^{-2} = 1.45 \times 0.135 = 0.195$

Power loss = $P_{ave}(\theta) - P_{ave}(\delta) = (1.45 - 0.195) \times 1\text{m}^2$

= 1.255W

Ques 7) A uniform plane wave propagating in a medium has

$E = 2e^{-\alpha z}\sin(10^8 t - \beta z)\ a_y$ V/m.

If the medium is characterised by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ mhos/m, find α, β, and H.

Ans: To determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \frac{10^{-9}}{36\pi}} = 3393 >> 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2}\right]^{1/2}$$

$= 61.4$

$\alpha = 61.4$ Np/m, $\quad \beta = 61.4$ rad/m

Also,

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3}\right]^{1/2}$$

$$= \sqrt{\frac{800\pi}{3}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \rightarrow \theta_\eta = 45° = \pi/4$$

Hence,

$$H = H_0 e^{-\alpha z}\sin\left(\omega t - \beta z - \frac{\pi}{4}\right)a_H$$

Where,

$a_H = a_k \times a_E = a_z \times a_y = -a_x$

and

$$H_0 = \frac{E_0}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$H = -69.1e^{-61.4z}\sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right)a_x \text{ mA/m}$$

Ques 8) In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $H = -0.1\cos(\omega t - z)\ a_x + 0.5\sin(\omega t - z)\ a_y$ A/m, calculate ε_r, ω and E.

Ans: In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}}$$

or $$\sqrt{\varepsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \rightarrow \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

or, $$\omega = \frac{\beta c}{2} = \frac{1(3\times10^8)}{2} = 1.5\times10^8 \text{ rad/s}$$

There are two ways to find the value of E and H,

Method 1: $H = H_1 + H_2$

Where,
$H_1 = -0.1 \cos(\omega t - z)\, a_x$ and $H_2 = 0.5 \sin(\omega t - z) a_y$ and the corresponding electric field

$$E = E_1 + E_2$$

Where,

$E_1 = E_{10} \cos(\omega t - z)\ a_{E_1}$ and $E_2 = E_{2_0} \sin(\omega t - z)\ a_{E_2}$.
Although H has components along a_x and a_y, it has no component along the direction of propagation; it is therefore a TEM wave.

For E_1,

$$a_{E_1} = -(a_k \times a_{H_1}) = -(a_z \times -a_x) = a_y$$
$$E_{1o} = \eta H_{1o} = 60\pi(0.1) = 6\pi$$

Hence
$$E_1 = 6\pi \cos(\omega t - z)\, a_y$$

For E_2,

$$a_{E_2} = -(a_k \times a_{H_2}) = -(a_z \times a_y) = a_x$$
$$E_{2_0} = \eta H_{20} = 60\pi(0.5) = 30\pi$$

Hence,
$$E_2 = 30\pi \sin(\omega t - z) a_x$$

Adding E_1 and E_2 gives E; that is,
$$E = 94.25 \sin(1.5\times10^8 t - z) a_x + 18.85 \cos(1.5\times10^8 t - z) a_y \text{ V/m}$$

Method 2: Apply Maxwell's equations directly,

$$\nabla\times H = \underset{\downarrow\, 0}{E} + \varepsilon\frac{\partial E}{\partial t} \Rightarrow E = \frac{1}{\varepsilon}\int \nabla\times H\, dt$$

Because $\sigma = 0$. But

$$\nabla\times H = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z} a_x + \frac{\partial H_x}{\partial z} a_y$$

$$= H_{2_0} \cos(\omega t - z) a_x + H_{1_0} \sin(\omega t - z) a_y$$

where, $H_{2_0} = -0.1$ and $H_{2_0} = 0.5$.

Hence

$$E = \frac{1}{\varepsilon}\int \nabla\times H\, dt = \frac{H_{2_0}}{\varepsilon\omega}\sin(\omega t - z)a_x - \frac{H_{1_0}}{\varepsilon\omega}\cos(\omega t - z)a_y$$

$= 94.25 \sin(\omega t - z)a_x + 18.85 \cos(\omega t - z)a_y$ V/m

Ques 9) A lossy dielectric has an intrinsic impedance of 200 $\angle 30°$ Ω at a particular frequency. If, a that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$H = 10e^{-\alpha x}\cos\left(\omega t - \frac{1}{2}x\right)a_y \text{ A/m}$$

Find E and α. Determine the skin depth and wave polarisation.

Ans: The given wave travels along a_x so that $a_k = a_x$; $a_H = a_y$,

so $\quad -a_E = a_k \times a_H = a_x \times a_y = a_z$

Or $\quad a_E = -a_z$

Also $H_o = 10$, so

$$\frac{E_0}{H_0} = \eta = 200\angle 30° = 200e^{j\pi/6} \rightarrow E_0 = 2000e^{j\pi/6}$$

Except for the amplitude and phase difference, E and H always have the same form. Hence

$$E = \text{Re}\,(2000e^{j\pi/6}e^{-\gamma x}e^{j\omega t}a_E)$$

or $$E = -2e^{-\alpha x}\cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right)a_z \text{ kV/m}$$

Knowing that $\beta = 1/2$, we need to determine α since

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

and $$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1}{\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1}\right]^{1/2}$$

But $\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60° = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2-1}{2+1}\right]^{1/2} = \frac{1}{\sqrt{3}}$$

or $\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887$ Np/m

and $\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464$m

The wave has an E_z component; hence it is polarised along the z-direction.

Ques 10) What do you mean by skin depth? Also define phase velocity and group velocity.

Ans: Skin Depth

As E (or H) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\gamma z}$ is called **media attenuation**. The distance δ, shown in **figure 6.3,** through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called **skin depth or penetration depth** of the medium; i.e.,

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

Or $$\delta = \frac{1}{\alpha} \quad .....(1)$$

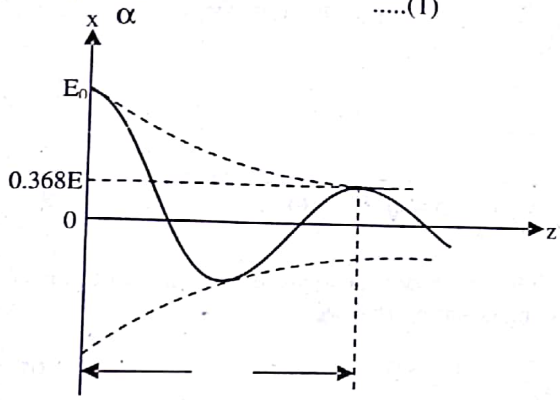


Figure 6.3: Illustration of Skin Depth

The skin depth is a measure of the depth to which an EM wave can penetrate the medium.

Equation (1) is generally valid for any material medium. For good conductors,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad .....(2)$$

Also for good conductors,

$$E = E_0 e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) a_x$$

Showing that δ measures the exponential damping of the wave as it travels through the conducting medium.

Phase Velocity

The phase velocity of a wave is the rate at which the phase of the wave propagates in space. This is the speed at which the phase of any one frequency component of the wave travels.

In terms of the cyclical frequency and wavelength, we have, $v_p = \lambda f$

$$\therefore \quad v_p = \frac{\omega}{\beta} = \lambda f = \frac{1}{\sqrt{\mu\varepsilon}}$$

Group Velocity

The velocity with which the overall shape of wave amplitude, known as the modulation or envelope of the wave, propagates through a medium is known as the **group velocity** of the wave. Group velocity is the velocity with which the energy propagates; hence, it is also known as **energy velocity**.

$$\therefore \quad v_g = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta}$$

Ques 11) What do you mean by surface resistance?

Ans: Surface Resistance

The surface or skin resistance R_s (in Ω/m^2) as the real part of the η for a good conductor. It is given by,

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the DC resistance for a unit length of the conductor having cross-sectional area 1 × δ. Thus for a given width w and length ℓ, the AC resistance is calculated using the familiar DC resistance and assuming a uniform current flow in the conductor of thicknessδ, that is,

$$R_{ac} = \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w}$$

Where, S = δw. For a conductor wire of radius a,

$$w = 2\pi a,$$

$$\text{So,} \quad \frac{R_{ac}}{R_{dc}} = \frac{\dfrac{\ell}{\sigma 2\pi a\delta}}{\dfrac{\ell}{\sigma\pi a^2}} = \frac{a}{2\delta}$$

Since δ << a, at high frequencies, this shows that R_{ac} is far greater than R_{dc}. In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is non-uniformly distributed over a thickness of 5δ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere. This is one more reason why δ is referred to as the skin depth.

TRANSMISSION LINES

Ques 12) Write the transmission line equations. Also derive the expression of characteristics impedance.

Or

Write the voltage and current equations for transmission line.

Or

What do you mean by transmission line? Derive the characteristics impedance of transmission lines.

Ans: Transmission Lines

A transmission line is a means of transfer of information from one point to another. Usually it consists of two conductors. It is used to connect a source to a load. The source may be a transmitter and the load may be a receiver.

The line parameters resistance (R), inductance (L), conductance (G), and capacitor (C) are not discrete or lumped but distributed as shown in **figure 6.4**. By this we mean that the parameters are uniformly distributed along the entire length of the line.

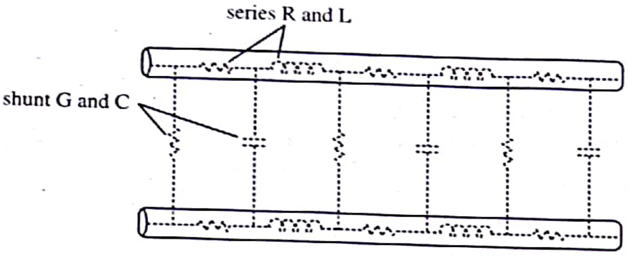


Figure 6.4: Distributed Parameters of a Two-Conductor Transmission Line

In the model of **figure 6.5**, we assume that the wave propagates along the +z-direction, from the generator to the load.

By applying Kirchhoff's voltage law to the outer loop of the circuit in **figure 6.5**, we obtain

$$V(z,\ t) = R\,\Delta z\, I(z, t) + L\,\Delta z \frac{\partial I(z,\ t)}{\partial t} + V(z+\Delta z,\ t)$$

or

$$\frac{V(z+\Delta z,\ t) - V(z,\ t)}{\Delta z} = R\ I(z,\ t) + L\frac{\partial I(z,\ t)}{\partial t} \quad .....(1)$$

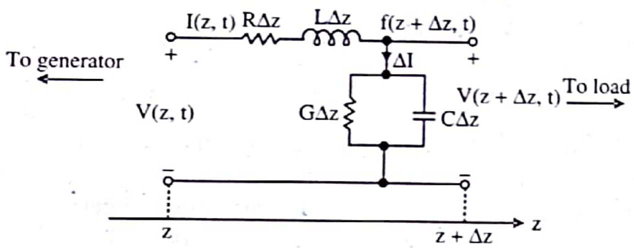


Figure 6.5: L-Type Equivalent Circuit Model of a Differential Length Δz of a Two-Conductor Transmission Line

Taking the limit of equation (1) as $\Delta z \to 0$ leads to

$$-\frac{\partial V(z,\ t)}{\partial z} = RI(z,\ t) + L\frac{\partial I(z,\ t)}{\partial t} \quad .....(2)$$

Similarly, applying Kirchoff's current law to the main node of the circuit in **figure 6.5** gives

$I(z, t) = I(z + \Delta z, t) + \Delta I$

$$= I(z+\Delta z,\ t) + G\,\Delta z\, V(z+\Delta z,\ t) + C\,\Delta z\frac{\partial V(z+\Delta z,\ t)}{\partial t}$$

$$-\frac{I(z+\Delta z,\ t) - I(z,\ t)}{\Delta z}$$

$$= G\,V(z+\Delta z,\ t) + C\frac{\partial V(z+\Delta z,\ t)}{\partial t} \quad .....(3)$$

As $\Delta z \to 0$, equation (3) becomes

$$-\frac{\partial I(z,\ t)}{\partial z} = G\,V(z,\ t) + C\frac{\partial V(z,\ t)}{\partial t} \quad .....(4)$$

If we assume harmonic time dependence so that

$V(z, t) = \mathrm{Re}[V_s(z)e^{j\omega t}]$ .....(5 a)

$I(z, t) = \mathrm{Re}[I_s(z)e^{j\omega t}]$ ....(5 b)

where $V_s(z)$ and $I_s(z)$ are the phasor forms of $V(z, t)$ and $I(z, t)$, respectively, equation (2) and (4) become

$$-\frac{dV_s}{dz} = (R + j\omega L)I_s \quad .....(6)$$

$$-\frac{dI_s}{dz} = (G + j\omega C)V_s \quad .....(7)$$

In the differential equation (6) and (7), V_s and I_s are coupled. To separate them, we take the second derivative of V_s in equation (6) and employ equation (7) so that we obtain

$$\frac{d^2V_s}{dz^2} = (R + j\omega L)(G + j\omega C)V_s$$

or

$$\frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \quad .....(8)$$

Where,

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}} \quad .....(9)$$

By taking the second derivative of I_s in equation (7) and employing equation (6), we get

$$\frac{d^2I_s}{dz^2} - \gamma^2 I_s = 0 \quad .....(10)$$

We notice that equation (8) and (10) are, respectively, the wave equations for voltage and current similar in form to the wave equations obtained for plane waves in equation (15) and (17). Thus, in our usual notations, γ in equation (9) is the propagation constant (in per meter), α is the attenuation constant (in nepers per meter or decibels[2] per meter), and β is the phase constant (in radians per meter). The wavelength λ and wave velocity u are, respectively, given by

$$\lambda = \frac{2\pi}{\beta} \quad .....(11)$$

$$\boxed{u = \frac{\omega}{\beta} = f\lambda} \quad ....(12)$$

The solutions of the linear homogeneous differential equations (8) and (10) namely,

$$V_s(z) = \underset{\to +z}{V_o^+ e^{-\gamma z}} + \underset{-z \leftarrow}{V_o^- e^{\gamma z}} \quad .....(13)$$

$$\text{and } I_s(z) = \underset{\to +z}{I_o^+ e^{-\gamma z}} + \underset{-z \leftarrow}{I_o^- e^{\gamma z}} \quad .....(14)$$

Where V_o^+, V_o^-, I_o^+, and I_o^- are wave amplitudes; the + and – signs, respectively, denote wave traveling along +z and –z-directions, as is also indicated by the arrows. Thus, we obtain the instantaneous expression for voltage as

$$V(z, t) = Re[V_s(z)e^{j\omega t}]$$
$$= V_o^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_o^- e^{\alpha z} \cos(\omega t + \beta z) \quad .....(15)$$

The characteristic impedance Z_o of the line is the ratio of positively traveling voltage wave to current wave at any point on the line.

Z_o is analogous to η, the intrinsic impedance of the medium of wave propagation. By substituting equation (13) and (14) into equation (6) and (7) and equating coefficients of terms $e^{\gamma z}$ and $e^{-\gamma z}$, we obtain

$$Z_o = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} \quad .....(16)$$

or
$$\boxed{Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_o + jX_o} \quad ....(17)$$

Ques 13) Explain the different types of waves in transmission lines.

Ans: Waves in Transmission Lines

There are two types of waves in transmission lines, they are as follows:

1) **Standing Waves:** Whenever there is a mismatch of impedance between transmission line and load, reflections will occur. If the incident signal is a continuous AC waveform, these reflections will mix with more of the oncoming incident waveform to produce stationary waveforms called **standing waves**.

   A wave confined to a given space in a medium and still produces a regular wave pattern that is readily discernible amidst the motion of the medium. **For example,** if an elastic rope is held end-to-end and vibrated at just the right frequency, a wave pattern would be produced that assumes the shape of a sine wave and is seen to change over time.

   The wave pattern is only produced when one end of the rope is vibrated at just the right frequency. When the proper frequency is used, the interference of the incident wave and the reflected wave occur in such a manner that there are specific points along the medium that appear to be standing still. Because the observed wave pattern is characterised by points that appear to be standing still, the pattern is often called a **standing wave pattern**. There are other points along the medium whose displacement changes over time, but in a regular manner. These points vibrate back and forth from a positive displacement to a negative displacement; the vibrations occur at regular time intervals such that the motion of the medium is regular and repeating. A pattern is readily observable.

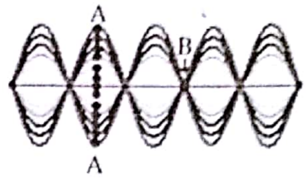


Figure 6.6

   The **figure 6.6** above depicts a standing wave pattern in a medium. A snapshot of the medium over time is depicted using various colours, point A on the medium moves from a maximum positive to a maximum negative displacement over time. The **figure 6.6** only shows one-half cycle of the motion of the standing wave pattern. The motion would continue and persist, with point A returning to the same maximum positive displacement and then continuing its back-and-forth vibration between the up to the down position, point B on the medium is a point that never moves. Point B is a point of no displacement. Such points are known as nodes. The standing wave pattern that is shown above is just one of many different patterns that could be produced within the rope.

2) **Travelling Waves:** These waves are the current and voltage waves which travel from the sending end of a transmission line to the other end. When the switch is closed at the transmission line's starting end, voltage will not appear instantaneously at the other end. This is caused by the transient behaviour of inductor and capacitors that are present in the transmission line. The transmission lines may not have physical inductor and capacitor elements but the effects of inductance and capacitance exists in a line.

   Therefore, when the switch is closed the voltage will build up gradually over the line conductors. This phenomenon is usually called as the voltage wave is travelling from transmission line's sending end to the other end. And similarly the gradual charging of the capacitances happens due to the associated current wave.

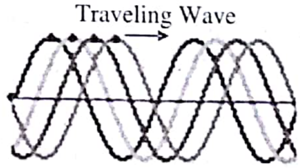


Figure 6.7: Crest is Seen to Move or Progress Across a Medium

   A mechanical wave is a disturbance that is created by a vibrating object and subsequently travels through a medium from one location to another, transporting energy as it moves. The mechanism by which a mechanical wave propagates itself through a medium involves particle interaction; one particle applies a push or pull on its adjacent neighbour, causing a displacement of that neighbour fro n the equilibrium or rest position.

As a wave is observed travelling through a medium, a crest is seen moving along from particle to particle. This crest is followed by a trough that is in turn followed by the next crest. In fact, one would observe a distinct wave pattern (in the form of a sine wave) travelling through the medium. This sine wave pattern continues to move in uninterrupted fashion until it encounters another wave along the medium or until it encounters a boundary with another medium. This type of wave pattern that is seen travelling through a medium is sometimes referred to as a **travelling wave**.

Ques 14) Determine the wave equation for loss less transmission lines

Ans: Wave Equation for Transmission Lines
A perfect transmission line will carry an electrical signal from one place to another in a fixed time, regardless of the rate at which the voltage changes. If applied a signal V (t) to one end of the transmission line, where t is time, the signal at the other end will be V (t – τ), where τ is a constant. One can model a real transmission line with a distributed inductance, capacitance, and resistance. To calculate τ, and so determine the circumstances under which τ will be constant. The following **figure 6.8** shows a small element of a transmission line:

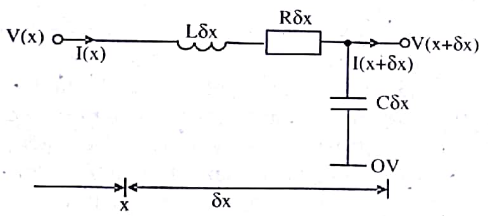


Figure 6.8: Transmission Line Element. Here C.L. and R as the capacitance, inductance, and resistance per unit length of line. Voltage with distance and time is V(x,t), current is I(x,t)

Our element is a short length, δx, of cable. Distance along the cable is x. Although the capacitance, inductance, and resistance of a transmission line are distributed and mingled with one another, one can lump them into three separate components in our infinitesimal element. As δx→0, our lumped model becomes a distributed model.

Consider the voltage across the inductor and resistor. At position x and time t a current I(x, t) passes through both of them in series.

The **voltage across inductive element** is given by,

$$V(x) - V(x+\delta x) = \frac{\partial I(x)}{\partial t} L\delta x + I(x)R\delta x$$

$$-\frac{\partial V}{\partial x}\delta x = \frac{\partial I}{\partial t} L\delta x + IR\delta x$$

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI \quad \text{...(1)}$$

The rate of change of voltage with x at a particular time is a function of the rate of change of current with time and the current itself.

The **current into capacitive element** is given by,

$$I(x) - I(x+\delta x) = \frac{\partial V(x+\partial x)}{\partial t} C\delta x$$

$$-\frac{\partial I(x)}{\partial x}\delta x = \frac{\partial V(x)}{\partial t} C\delta x + \frac{\partial^2 V(x)}{\partial x \partial t}\delta x C\delta x$$

$$\lim_{\delta x \to 0} \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \quad \text{...(2)}$$

The rate of change of current with x at a particular time is proportional to the rate of change of voltage with time. Let us differentiate equation (1) with respect to x and equation (2) with respect to t. Let us use the resulting equations to eliminate terms in I.

The **transmission line equations** is given by,

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial I}{\partial x \partial t} - R\frac{\partial I}{\partial x}$$

$$\frac{\partial^2 I}{\partial x \partial t} = -C\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} - RC\frac{\partial V}{\partial t}$$

If arrived at a partial differential equation in V. Let us assume R is zero, the second derivative in x being proportional to the second derivative in t. These are the conditions under which a sinusoidal wave will propagate without distortion or attenuation. Consider a sinusoid of frequency f = ω/2π, as shown below.

The **propagating sine wave** is given by,

$$V(x,t) = a\sin\left(\omega t - w\sqrt{LC}x\right)$$

$$\frac{\partial^2 V}{\partial x^2} = a\omega^2 LC\sin\left(\omega t - \omega\sqrt{LC}x\right)$$

$$\frac{\partial^2 V}{\partial t^2} = a\omega^2 \sin\left(\omega t - \omega\sqrt{LC}x\right)$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$$

The sinusoidal wave has the unique property that its derivatives have the same shape as the original. There is some scaling of the amplitude of the waveform as differentiation takes place, and it is this scaling that constrains the solution to our transmission line equation.

If t = √(LC) x, the movement of the positive zero-crossing of the sinusoid (the value of sine when its angle is zero). Therefore dx/dt = 1/√(LC). The velocity of the sine wave is 1/√(LC). Provided that L and C remain constant with ω, the velocity of all sine waves will be the same.

Let V (0, t) denote the voltage at position zero and time t. If represent our input V(0, t) as a sum of sinusoids using a Fourier transform, all these sinusoids will propagate along the transmission line at the same speed, so that their sum will remain undistorted as it propagates, and it will have our ideal transmission line: $V(x, t) = V(0, t - \tau)$ with $\tau = x\sqrt{LC}$.

Ques 15) Derive the expression of input impedance. Also write its value for lossless line.

Or

Discuss the voltage and current distribution of a line terminated with load.

Or

Derive the expression of reflection coefficient and Power for transmission line.

Or

Discuss the reflection coefficient for short, open and matched transmission line.

Ans: Input Impedance

Consider a transmission line of length ℓ, characterised by γ and Z_o, connected to a load Z_L as shown in **figure 6.9.** Looking into the line, the generator sees the line with the load as input impedance Z_{in}.

It is our intention in this section to determine the input impedance, the standing wave ratio (SWR), and the power flow on the line.

Let the transmission line extend from $z = 0$ at the generator to $z = \ell$ at the load. First of all, we need the voltage and current waves, that is

$$V_s(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \quad .....(1)$$

$$I_s(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{\gamma z} \quad .....(2)$$

To find V_o^+ and V_o^-, the terminal conditions must be given. **For example,** if we are given the conditions at the input, say

$$V_o = V(z = 0),\ I_o = I(z = 0) \quad .....(3)$$

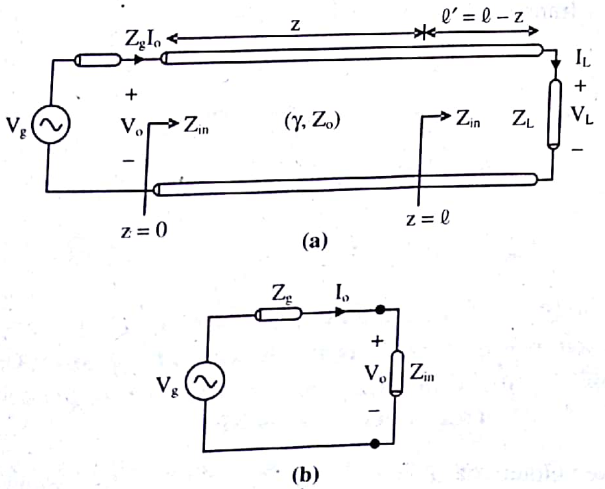


Figure 6.9: (a) Input Impedance Due to a Line Terminated by a Load; (b) Equivalent Circuit for Finding V_o and I_o in Terms of Z_{in} at the Input

substituting these into equation (1) and (2) results in

$$V_o^+ = \frac{1}{2}(V_o + Z_o I_o) \quad .....(4a)$$

$$V_o^- = \frac{1}{2}(V_o + Z_o I_o) \quad .....(4b)$$

If the input impedance at the input terminals is Z_{in}, the input voltage V_o and the input current I_o are easily obtained from **figure 6.9 (b)** as:

$$V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g,\ I_o = \frac{V_g}{Z_{in} + Z_g} \quad .....(5)$$

On the other hand, if we are given the conditions at the load, say

$$V_L = V(z = \ell),\ I_L = I(z = \ell) \quad .....(6)$$

Substituting these into equation (1) and (2) gives,

$$V_o^+ = \frac{1}{2}(V_L + Z_o I_L)e^{\gamma \ell} \quad .....(7a)$$

$$V_o^- = \frac{1}{2}(V_L - Z_o I_L)e^{-\gamma \ell} \quad .....(7b)$$

Next, we determine the input impedance $Z_{in} = V_s(z)/I_s(z)$ at any point on the line. At the generator, e.g., equation (1) and (2) yield

$$Z_{in} = \frac{V_s(z)}{I_s(z)} = \frac{Z_o(V_o^+ + V_o^-)}{V_o^+ - V_o^-} \quad .....(8)$$

Substituting equation 7(a),(b) into (8) and utilising the fact that

$$\frac{e^{\gamma\ell} + e^{-\gamma\ell}}{2} = \cosh \gamma\ell,\ \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{2} = \sinh \gamma\ell, \quad .....(9a)$$

or $$\tanh \gamma\ell = \frac{\sinh \gamma\ell}{\cosh \gamma\ell} = \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{e^{\gamma\ell} + e^{-\gamma\ell}} \quad .....(9b)$$

We get

$$\boxed{Z_{in} = Z_o\left[\frac{Z_L + Z_o \tanh \gamma\ell}{Z_o + Z_L \tanh \gamma\ell}\right]} \text{ (Lossy) .....(10)}$$

Although equation (10) has been derived for the input impedance Z_{in} at the generation end, it is a general expression for finding Z_{in} at any point on the line. To find Z_{in} at a distance ℓ' from the load as in **figure 6.9 (a)**, we replace ℓ by ℓ'. A formula for calculating the hyperbolic tangent of a complex number, required in equation (10).

For a lossless line, $\gamma = j\beta$, $\tanh j\beta\ell = j \tan \beta\ell$, and $Z_o = R_o$, so equation (10) becomes,

$$\boxed{Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell}\right]} \text{ (Lossless).....(11)}$$

showing that the input impedance varies periodically with distance ℓ from the load. The quantity $\beta\ell$ in equation (11) is usually referred to as the electrical length of the line and can be expressed in degrees or radians.

Reflection Coefficient

We now define Γ_L as the voltage reflection coefficient (at the load). Γ_L is the ratio of the voltage reflection wave to the incident wave at the load, that is,

$$\Gamma_L = \frac{V_o^- e^{\gamma\ell}}{V_o^+ e^{-\gamma\ell}} \quad .....(12)$$

Substituting V_o^- and V_o^+ in equation 7(a) and (b) into equation (12) and incorporating $V_L = Z_L I_L$ gives,

$$\boxed{\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}} \quad .....(13)$$

The voltage reflection coefficient at any point on the line is the ratio of the magnitude of the reflected voltage wave to that of the incident wave.
That is,

$$\Gamma(z) = \frac{V_o^- e^{\gamma z}}{V_o^+ e^{-\gamma z}} = \frac{V_o^-}{V_o^+} e^{2\gamma z}$$

But $z = \ell - \ell'$. Substituting and combining with equation (13), we get

$$\Gamma(z) = \frac{V_o^-}{V_o^+} e^{2\gamma\ell} e^{-2\gamma\ell'} = \Gamma_L e^{-2\gamma\ell'} \quad .... (14)$$

Ques 16) What is the input impedance of 55Ω lossless transmission line having length 0.2λ, if the load is a short circuit?

Ans: It is given that $\ell = 0.2\lambda$

$Z_o = 55\Omega$ and

$Z_i = 0$ (for a short-circuit line)

Z_i (for a lossless line) is given as

$$Z_i = Z_o \left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jL_L \tan\beta\ell} \right]$$

Electric length

$$\beta\ell = \left(\frac{2\pi}{\lambda}\right) \times 0.2\lambda$$
$$= 0.4\pi$$

Thus,

$$Z_i = jZ_o \tan\beta\ell \quad \text{or}$$
$$= j \times 55 \times \tan 0.4\pi \quad \text{or}$$
$$= j \times 55 \times 0.0219 \quad \text{or}$$
$$Z_i = j\ 1.20\Omega$$

Ques 17) A 55Ω lossless transmission line has a length of 0.3λ. If it is terminated in an open circuit, find the input impedance.

Ans: It is given that

$\ell = 0.3\lambda$ and

$Z_o = 55\Omega$

Electric length,

$$(\beta\ell) = \frac{2\pi}{\lambda} \times 0 \times 3\lambda \quad \text{or} = 0.6\pi$$

Input impedance Z_{in} is given as,

$$Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell} \right]$$

but $Z_L = \infty$ (for a short-circuit line).

Thus,

$$Z_{in} = Z_o \left[\frac{1}{j\tan\beta\ell} \right] \quad \text{or}$$

$$Z_{in} = -jZ_o \cot\beta\ell \quad \text{or}$$
$$Z_{in} = -j \times 55 \times \cot(0.6\pi) \quad \text{or}$$
$$Z_{in} = -1671.19\,j\Omega$$

Ques 18) A transmission line has a characteristic impedance of 40Ω. It is terminated at a reactance of j25Ω. Find the input impedance of a section, which is 50m long at a frequency of 150MHz.

Ans: It is given that $Z_o = 40\Omega$ and $Z_L = j25\Omega$. Also, frequency (f) = 150MHz and

$$\ell = 50\text{cm}$$
$$= \frac{1}{2}\text{m}$$

Wavelength

$$\lambda = \left(\frac{v}{f}\right)$$
$$= \left(\frac{3 \times 10^8}{150 \times 10^6}\right) \quad \text{or}$$
$$= 2\text{m}$$

Thus,

$$\ell = \lambda/4$$

It is a quarter-wave transmission line. Z_{in} in a quarter-wave transmission line is obtained as

$$Z_{in} = \frac{Z_0^2}{Z_L}$$

or $$Z_{in} = \frac{(40)^2}{j25}$$

or $$Z_{in} = -64j\Omega$$

Ques 19) A transmission line has a characteristic impedance of 60Ω. It is terminated at a resistance of j40Ω. Find the input impedance of a section, which is 25cm long at a frequency of 300MHz.

Ans: Given that $Z_0 = 60\Omega$ and $Z_L = j40\Omega$. Also, frequency (f) = 200MHz.

$$\lambda = \left(\frac{v}{f}\right) = \frac{3 \times 10^8}{300 \times 10^6} = 1\text{m}$$

$$\ell = \frac{25}{100}\text{m} = 0.25\text{m}$$

Since $\ell = \frac{\lambda}{4}$

Thus it is a quarter-wave transmission line and Z_{in} in a quarter-wave transmission line is obtained as

$$Z_{in} = \frac{Z_o^2}{Z_L} \quad \text{or}$$

$$Z_{in} = \frac{(60)^2}{j40} = -90j\Omega$$

$$Z_{in} = -90j\Omega$$

Ques 20) What do you mean by VSWR? Compute the relation between reflection coefficient and VSWR.

Ans: Voltage Standing-Wave Ratio (VSWR)

A useful and frequently used concept related to the reflection coefficient is the voltage standing-wave ratio, or VSWR. The VSWR is the ratio of the maximal to minimal voltage along the line.

Relation between Reflection Coefficient and Voltage Standing Wave Ratio (VSWR)

VSWR is defined as $VSWR = \frac{V_{max}}{V_{min}} = \left(\frac{1+|\Gamma|}{1-|\Gamma|}\right)$

Proof

$$V_{max} = |V_i| + |V_r| = |V_i|\,[1+|\Gamma|]$$

Similarly,

$$V_{min} = |V_i| - |V_r| = |V_i|(1-|\Gamma|)$$

$$VSWR = S = \frac{V_i(1+|\Gamma|)}{V_i(1-|\Gamma|)}$$

$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

S ranges between 1 and ∞ or $1<S<\infty$

We can also write $|\Gamma|$ as

$$|\Gamma| = \frac{S-1}{S+1}$$

Ques 21) A certain R.F. transmission line is terminated in pure resistive load. The characteristic impedance of the line is 1200Ω and the reflection coefficient was observed to be 0.2. Calculate the terminating load, which is less than characteristic impedance.

Ans: for resistive load, $S = \frac{Z_o}{R_R} = \frac{1200}{R_R}$ and reflection coefficient in terms of VSWR is given as,

$$\Gamma = \frac{s-1}{s+1} = 0.2$$

$$\therefore \quad \Gamma = \frac{\left(\frac{1200}{R_R} - 1\right)}{\left(\frac{1200}{R_R} + 1\right)} = 0.2$$

Simplifying for R_R, we get:

$R_R = 800\Omega$

Ques 22) Calculate standing wave ratio and reflection coefficient on a line having Z_o = 300Ω and terminated in $Z_R = 300 + j400$.

Ans: The reflection coefficient is given by,

$$\Gamma = \frac{Z_R - Z_o}{Z_R + Z_o} = \frac{(300+j400)-(300)}{(300+j400)+(300)} = \frac{j400}{600+j400}$$

$$\therefore \quad \Gamma = \frac{400\angle 90°}{721.11\angle 33.69°}$$

$$\therefore \quad \Gamma = 0.5547\angle 56.31°$$

Ques 23) What is an impedance matching using stub line? Also describe the single stub matching using analytical method.

Ans: Impedance Matching Using Stub Lines

For maximum power transfer, we know that the source and load impedances should match. However, in the case of long transmission lines, these impedances must be equal to the characteristic impedance Z_0 of the line. In many situations, the source (such as a transmitter) and the load (such as an antenna) connected by the long transmission line never match in impedance values, In such situations, maximum power does not get transferred from the transmitter to the antenna.

Since stub lines can transform and match impedances. So, to match impedances in long transmission lines, stub lines of suitable lengths can be employed. There are two methods of stub-line matching using analytical method and smith chart.

Single-Stub Impedance Matching using Analytical Method

The stub line acts as a reactance and resonates with the load. This idea is elaborated further as given below:

As voltage standing-wave ratio S is given by

$$S = \frac{Z_s}{R_0} = \frac{R_s}{R_0} + j\frac{X_s}{R_0} \quad \text{.... (1)}$$

Where Z_s (= R_s +jX_s) is the input impedance, and R_0 = characteristic resistance of the line.

Therefore from equation (1),

Input resistance $R_S = SR_0$ (real part) .... (2) and

Input reactance $X_S = SR_0$ (imaginary part).... (3)

Now, at maximum voltage, the imaginary part $X_S = 0$. Also, the input conductance G_s is minimum, and is given by

$$G_{S\min} = \frac{1}{R_S} = \frac{1}{SR_0} = \frac{1}{R_0}\left(\frac{1-\Gamma}{1+\Gamma}\right) = G_0\left(\frac{1-\Gamma}{1+\Gamma}\right) \quad \text{.... (4)}$$

In equation(4), we have used the relations

$$S = \frac{\Gamma+1}{\Gamma-1} = \frac{1+\Gamma}{1-\Gamma} \quad \text{.... (5)}$$

$$G_0 = \frac{1}{R_0} \quad \text{.... (6)}$$

One of the means of achieving impedance transformation is to use a stub line of suitable length ℓ, as shown in **figure 6.10**, at a distance x from the load-end such that the line is brought to resonance.

In a similar manner, we find that at a voltage minimum also $X_S = 0$, which gives the expression for conductance as:

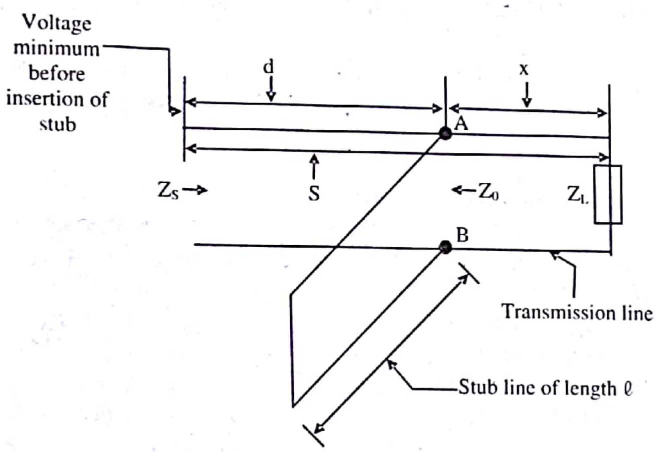


Figure 6.10: Single-Stub Impedance Matching

$$G_{S\max} = \frac{S}{R_S} = \frac{1}{R_0}\left(\frac{1+\Gamma}{1-\Gamma}\right) \quad \text{.... (7)}$$

In between the maximum and minimum values of conductance, we get the expression for the input admittance as

$$Y_S = G_0 \pm jB \quad \text{.... (8)}$$

Where, B is susceptance of the line. We insert our stub line in between the maximum and the minimum values of conductance so that it will act as an anti-susceptance term to cancel the B term in equation (8) at resonance. That is, at resonance

$$Y_S = G_0 \pm jB = G_0 \quad \text{.... (9)}$$

Where the first ±jB term is due to the transmission line, and the second (cancelling) ∓jB term is due to the stub line.

Ques 24) Discuss the electromagnetic interferences and also give its classifications.

Or

Discuss the electromagnetic interferences and also write the sources of EMI

Ans: Electromagnetic Interference

Electromagnetic interference is the degradation in the performance of a device due to the fields making up the electromagnetic environment.

Interference occurs if the received energy causes the receptor to function in unwanted manner. Whether the receiver is functioning in wanted or unwanted manner, depends on the coupling path as well as the source and victim. The medium is to be made as inefficient as possible.

The basic elements of EMI are shown in **figure 6.11** below

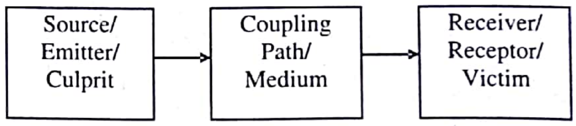


Figure 6.11: Basic Elements of EMI Situation

Classification of EMI

The classification of EMI is shown in **figure 6.12** below.

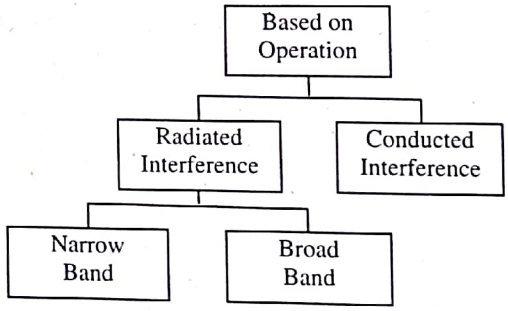


Figure 6.12: Classification of EMI

There are two types of interferences, they are as follows:

1) **Radiated Interferences:** The radiated interference is of two types they are a follows:

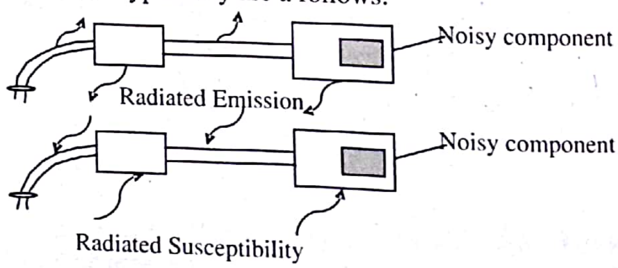


Figure: Radiated Interference

i) **Narrow Band:** Narrow band interference usually arises from intentional transmissions such as radio and T.V. stations, pager transmitters, cell phones, etc. It is a high frequency operation.

For example, proximity effect

ii) **Broad Band:** Broad band interference usually comes from incidental radio frequency emitters. These includes electric power transmission lines, electric motors etc. It is a low frequency operation.

For example, skin effect

2) **Conducted Interference:** Conducted electromagnetic interference is caused by the physical contact of the conductors as opposed to radiated EMI, which is caused by induction (without physical contact of the conductors).

Electromagnetic disturbances in the EM field of a conductor will no longer be confined to the surface of the conductor and will radiate away from it.

This persists in all conductors and mutual inductance between two radiated electromagnetic fields will result in EMI.

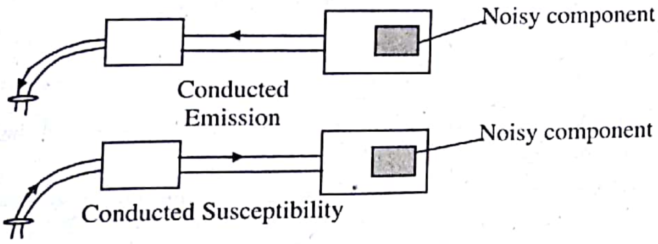


Figure 6.13: Conducted Interference

Sources of EMI

The sources of EMI can be broadly classified into two groups:

1) **Natural Sources of EMI: For example,** lightning.
2) **Manmade Sources of EMI: For example,** commercial radio and telephone communications.

In specific we can classify as:

1) **Functional:** EMI can originate from any source designed to generate electromagnetic energy and which may create interference as a normal part of its operation.
2) **Incidental:** EMI can originate from manmade sources. These sources are not designed specifically to generate electromagnetic energy but which do infact cause interference.
3) **Natural:** EMI can be caused by natural phenomena, such as electrical storms rain particles, solar and interstellar radiation.

Ques 25) Discuss the electromagnetic compatibility and also explain the need for EMC standards.

Or

Explain electromagnetic compatibility. Also write the types of EMC standards and state the advantages of EMC standards.

Ans: Electromagnetic Compatibility

Electromagnetic Compatibility (EMC) is the branch of electrical science which studies the unintentional generation, propagation and reception of electromagnetic energy with reference to the unwanted effects (Electromagnetic interference, or EMI) that such energy may induce.

The goal of EMC is the correct operation, in the same electromagnetic environment, of different equipment which uses electromagnetic phenomena, and the avoidance of any interference effects.

A system is said to be electromagnetically compatible if:

1) It does not cause interference with other system.
2) It is not susceptible to emissions from other systems.
3) It does not cause interference with itself.

EMI is a phenomenon while EMC is an equipment characteristic or a property not to generate EMI above a certain limit and not to be affected or disturbed by EMI.

Electromagnetic compatibility is achieved when a device functions satisfactorily without introducing intolerable disturbances to the electromagnetic environment.

The statement "Live and let live" is the best way to describe EMC.

Need for EMC Standards

The EMC standards are required for trouble-free co-existence and to ensure satisfactory operation. They are also required to provide compatibility between electrical, electronic, computer, control and other systems. Standards are required as manufacturer–user interaction and user's knowledge on EMI are limited. They are also required for establishing harmonised standards to reduce international trade barriers and to improve product reliability and life of the product.

Types of EMC Standards

These are of two types:

1) **Military Standards:** Military EMC standards are made in order to ensure system-to-system compatibility in the real time military environment. Military standards are more stringent than civilian standards. Most of the military standards are broadly based on MIL-STD 461 and 462.
2) **Civilian Standards:** The civilian EMC standards are applicable for equipment used for commercial, industrial and domestic applications. The emission standards are specified to protect the broadcast services from interference.

Advantages of EMC Standards

The advantages are as follows:

1) Compatibility, reliability and maintainability are increased.
2) Design safety margin is provided.
3) The equipment operates in EMI scenario satisfactorily.
4) Product life and profits are increased.

MODEL PAPER

ELECTROMAGNETICS

B. TECH. SIXTH SEMESTER EXAMINATION

Time: 3 Hours **Max. Marks: 100**

Part-A

Note: Attempt all question . Each question carrying 5 marks.

Ques 1) Transform Vector $A = y\hat{a}_x + (x+z)\hat{a}_y$ into spherical coordinates system. Also evaluate at p (–2, 6, 3).

Ques 2) Find the electric field intensity due to infinite line charged wire (line charged).

Ques 3) The mean radius of a circular coil of 50 turns of fine wire is 8.0 cm, It carries a current of 3.0 A. The coil is located on the y–z plane in air. Find the magnetic field intensity vector at P(20 cm.0,0).

Ques 4)Explain briefly the different types of polarisation in dielectrics.

Ques 5) A uniform plane wave at frequency of 300MHz travels in vacuum along +y direction. The electric field of the wave at some instant is given as $E = 3\hat{x} + 5\hat{z}$. Find the phase constant of the wave and also the vector magnetic field.

Ques 6) A uniform plane wave travelling along positive z direction in air strikes normally on the surface of a dielectric with $\mu = \mu_0$ and $\varepsilon = 6.25\varepsilon_0$. The amplitude of electric field of the incident wave is 10 V/m. Calculate the amplitudes of electric field intensities associated with the reflected and transmitted waves, assuming that the dielectric extends to infinity. Also, calculate the power per unit area carried by each wave.

Ques 7) In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $H = -0.1\cos(\omega t - z)\, a_x + 0.5\sin(\omega t - z)\, a_y$ A/m, calculate ε_r, ω and E.

Ques8) What do you mean by skin depth? Also define phase velocity and group velocity.

Part-B (Modules I and II)

Note: Answer Two full questions. Each question carrying 10 marks.

Ques 9 a) Determine the curl of the following vector fields:

1) $\vec{F} = x^2 y\hat{a}_x + y^2 z\hat{a}_y - 2xz\hat{a}_z$

2) $\vec{A} = r^2 \sin\phi\,\hat{a}_r + r\cos^2\phi\,\hat{a}_\phi + z\tan\phi\,\hat{a}_z$

3) $\vec{V} = \dfrac{\sin\phi}{\rho^2}\hat{a}_\rho - \dfrac{\cos\phi}{\rho^2}\hat{a}_\phi$

b) State and prove stokes' theorem.

Ques 10 a) Prove the electric field vector E = (grad v), where v is a scalar potential field.

b)Derive the expressions of capacitance for a coaxial cable.

Ques 11a) The linear charge density of an infinite line charge located along the axis of a cylinder of radius r is 8 nC/m. The axis of the cylinder is the z axis of cylindrical coordinates. Find the electric flux crossing a part of the cylinder defined by $30^0 \le \phi \le 60^0$ and $0 \le z \le 3.6$ m.

b)Give the statement of Gauss law and define its value for integral and point form.

c) Give that $\overline{A} = 30e^{-r}\overline{a}_r - 2z\,\overline{a}_z$ in the cylindrical co-ordinates. Evaluate both sides of the divergence theorem for the volume enclosed by r = 2, z = 0 and z = 5.

Part-C (Modules III and IV)

Note: Answer Two full questions. Each question carrying 10 marks.

Ques 12 a)State and explain Biot-savart's law.

b)Derive the magnetic field intensity on the axis of a circular loop carrying current I.

c)Solve the following:

1) A radial field, $\vec{H} = \dfrac{2.39\times10^6}{r}\cos\phi\,\hat{a}_r$ A/m, exists in free space. Find the magnetic flux ϕ crossing the surface defined by $-\pi/4 \le \phi \le \pi/4, 0 \le z \le 1$m.

2) Compute the total magnetic flux ϕ crossing the z = 0 plane in cylindrical coordinates for $r \le 5\times10^{-2}$ m if, $\vec{B} = \dfrac{0.2}{r}\sin^2\phi\,\hat{a}_r$(T).

Ques 13 a) Determine the boundary conditions of magnetic field from Maxwell's laws.

b) There are two homogenous, linear and isotropic, media with interface at x = 0.x < 0 describes medium 1 (μ_{r1} = 4). x > 0 describes medium 2 (μ_{r2} = 10). Magnetic field medium 1 is $(30\overline{a}_x - 80\overline{a}_y + 70\overline{a}_z)$ (A/m). Find the magnetic field in medium 2. Also, find the magnetic flux density in medium 1.

Ques 14 a) Derive the expression for energy stored in an electric field.

b)Derive the magnetic field intensity on the axis of a rectangular loop carrying current I.

Part-D (Modules V and VI)

Note: Answer Two full questions. Each question carrying 10 marks.

Ques 15a) Explain the reflection of plane wave for the normal incidence. Discuss about reflection and transmission coefficient for E and H.

b) State and derive Poynting Vector theorem and also express it in complex form.

c) For a wave travelling in air, the electric field is given by $\overline{E}$ = 6 cos ($\omega t - \beta z$) $\overline{a}_x$ at frequency 10MHz. Calculate:

1) β, 2) $\overline{H}$, and 3) Average Poynting vector

Ques 16 a) Discuss the wave propagation in loss dielectrics. Also determine the E and H for the loss dielectrics.

b) A lossy dielectric has an intrinsic impedance of 200 $\angle 30°$ Ω at a particular frequency. If, a that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$H = 10e^{-\alpha x}\cos\left(\omega t - \frac{1}{2}x\right)a_y \text{ A/m}$$

Find E and α. Determine the skin depth and wave polarisation.

Ques 17 a) What do you mean by transmission line? Derive the characteristics impedance of transmission lines.

b) A 55Ω lossless transmission line has a length of 0.3λ. If it is terminated in an open circuit, find the input impedance.

c) Discuss and derive the wave equations in phasor form.